Homeomorphisms of the annulus with a transitive lift

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Abstract

Let f be a homeomorphism of the closed annulus A that preserves orientation, boundary components and that has a lift $\tilde{f}$ to the infinite strip $\tilde{A}$ which is transitive. We show that, if the rotation number of both boundary components of A is strictly positive, then there exists a closed nonempty connected set $\Gamma \subset \tilde{A}$ such that $\Gamma \subset]-\infty, 0] \times [0,1]$, Γ is unlimited, the projection of Γ to A is dense, $\Gamma - (1,0) \subset \Gamma$ and $\tilde{f}(\Gamma) \subset \Gamma$. Also, if p_1 is the projection in the first coordinate in $\tilde{A}$, then there exists $d > 0$ such that, for any $\tilde{z} \in \Gamma$,

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\tilde{z})) - p_1(\tilde{z})}{n} < -d.$$

In particular, using a result of Franks, we show that the rotation set of any homeomorphism of the annulus that preserves orientation, boundary components, which has a transitive lift without fixed points in the boundary is an interval with 0 in its interior.

Key words: closed connected sets, order, transitivity, stable sets, periodic orbits, compactification

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1 Introduction and statements of the main results

In this paper we consider homeomorphisms of the closed annulus $A = S^1 \times [0,1]$, which preserve orientation and the boundary components. Any lift of f to the universal cover of the annulus $\widetilde{A} = \mathbb{R} \times [0,1]$, is denoted by $\widetilde{f}$, a homeomorphism which satisfies $\widetilde{f}(\widetilde{x}+1, \widetilde{y}) = \widetilde{f}(\widetilde{x}, \widetilde{y}) + (1,0)$ for all $(\widetilde{x}, \widetilde{y}) \in \widetilde{A}$. We study properties such homeomorphisms when they have a particular lift $\widetilde{f}$ which is transitive.

Our results do not assume the existence of invariant measures of any type for f, yet the importance of studying consequences of transitivity for such mappings is underlined by the results of [3], which imply that $\mathcal{C}^1$-generically an area preserving diffeomorphism f of the closed annulus is transitive.

In order to motivate our hypotheses a little more, let us define, for any homeomorphism $f : A \rightarrow A$ which preserves orientation and the boundary components and for any Borel probability f-invariant measure μ, an invariant called the rotation number of μ, as follows:

Let $p_1 : \widetilde{A} \rightarrow \mathbb{R}$ be the projection on the first coordinate and let $p : \widetilde{A} \rightarrow A$ be the covering mapping. Fixed f and $\widetilde{f}$, the displacement function $\phi : A \rightarrow \mathbb{R}$ is defined as

$$\phi(x,y) = p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) - \widetilde{x}, \tag{1}$$

for any $(\widetilde{x}, \widetilde{y}) \in p^{-1}(x,y)$. The rotation number of μ is then given by

$$\rho(\mu) = \int_A \phi(x,y) d\mu.$$

The importance of this definition becomes clear by Birkhoff's ergodic theorem, which states that, for μ almost every (x,y) in the annulus and for any $(\widetilde{x}, \widetilde{y}) \in p^{-1}(x,y)$,

$$\rho(x,y) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ f^i(x,y) = \lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(\widetilde{x}, \widetilde{y}) - \widetilde{x}}{n},$$

exists and

$$\int_A \rho(x,y) d\mu = \rho(\mu).$$

Moreover, if f is ergodic with respect to μ, then $\rho(x,y)$ is constant μ-almost everywhere.

Following the usual definition (see [1]), we refer to the set of area, orientation and boundary components preserving homeomorphisms of the annulus, which satisfy $\rho(Leb) = 0$ for a certain fixed lift $\widetilde{f}$, by rotationless homeomorphisms. Every time we say that f is a rotationless homeomorphism, a special lift $\widetilde{f}$ is fixed; the one used to define ϕ.

In [2] it is proved that transitivity of $\widetilde{f}$ holds for a residual subset of rotationless homeomorphisms of the annulus and the results in [3] suggest that the same statement may hold in the $\mathcal{C}^1$ topology.

Our original motivation in this setting was to study a problem posed by P. Boyland, which will be explained below.

Given a rotationless homeomorphism of the annulus f, by a result of Franks (see [4]), if there are 2 f-invariant probability measures μ_1 and μ_2 with $\rho(\mu_1) < \rho(\mu_2)$, then for every rational $\rho(\mu_1) < \frac{p}{q} < \rho(\mu_2)$, there exists a q-periodic orbit for f with this rotation number. So, suppose there exists a measure with positive rotation number. By a classical result (a version of the Conley-Zehnder theorem to the annulus) there must be fixed points with zero rotation number, so Boyland's question is: Is it true that in the above situation there must be orbits with negative rotation number? This is a very difficult problem, which we did not solve in full generality. We considered the following situation:

Suppose f is an orientation and boundary components preserving homeomorphism of the annulus which has a transitive lift $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ (one with a dense orbit). We denote the set of such mappings by $Hom_+^{trans}(A)$. So every time we say $f \in Hom_+^{trans}(A)$ and refer to a lift $\widetilde{f}$ of f, we are always considering a transitive lift (maybe f has more then one transitive lift, we choose any of them). Our results are the following:

Theorem 1 *: If $f \in Hom_+^{trans}(A)$ and the rotation number of $(f, \widetilde{f})$ restricted to the boundaries of the annulus, $\rho(\widetilde{f} \mid_{\mathbb{R}\times\{i\}})$, is strictly positive for $i = 0, 1$, then there exists a closed set*

$$B^- \subset]-\infty, 0] \times [0,1], B^- \cap \{0\} \times [0,1] \neq \emptyset,$$

such that every connected component of B^- is unlimited to the left, $\widetilde{f}(B^-) \subset B^-$ and $p(B^-)$ is dense in the annulus. Moreover, B^- is the subset of

$$B = \bigcap_{n \leq 0} \widetilde{f}^n(]-\infty, 0] \times [0,1])$$

which has only unlimited connected components.

Apart from the properties described in theorem 1, we also show:

Theorem 2 *:*

$$\omega(B^-) = \bigcap_{n=0}^{\infty} \overline{\left(\bigcup_{i=n}^{\infty} \widetilde{f}^i(B^-) \right)} = \emptyset.$$

Thus, iterates of B^- by $\tilde{f}$ converge to left end of $\widetilde{A}$. The properties of B^- allow us to extend this theorem and obtain a stronger result:

Theorem 3 *: There exists a real number $\rho^+(B^-) < 0$ such that, if $\widetilde{z} \in B^-$, then*

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})}{n} \leq \rho^+(B^-) < 0.$$

The last theorem shows that all points in B^- have a "minimum negative velocity" in the strip $\tilde{A}$.

As $\widetilde{f}$ has a dense orbit, so does f and thus every point in the annulus A is non-wandering for f. In this way, theorem 3 together with Franks version of the Poincaré-Birkoff's theorem from [5] implies the following:

Corollary 1 *: Let $f \in Hom_{+}^{trans}(A)$. If $\tilde{f}$ does not have fixed points in the boundary of $\widetilde{A}$, then the rotation set is an interval with 0 in its interior.*

Another important consequence of theorem 3 is that, even though there are points with rotation number in $]\rho^+(B^-),0[$, they do not belong to B^-. In particular, if such points have unstable manifolds unbounded to the left, they must also be unbounded to the right.

Our next results, which are corollaries of the methods used to prove theorems 1 and 2, give more information on the structure of B^-. Their hypothesis are the same, namely, $f \in Hom_{+}^{trans}(A)$ and $\rho(\widetilde{f}\mid_{\mathbb{R}\times\{i\}}) > 0$ for $i = 0, 1$.

Theorem 4 *: There exists a connected component Γ of B^-, such that $p(\Gamma)$ is dense in the annulus, $\widetilde{f}(\Gamma) \subset \Gamma$, and there is a positive integer k such that $\widetilde{f}^{-1}(\Gamma) \subset \Gamma + (k, 0)$, so $f(p(\Gamma)) = p(\Gamma)$.*

Theorem 5 *: If Γ is a connected component of B^-, then $\Gamma - (1, 0) \subset \Gamma$.*

Theorems 3, 4 and 5 above have an interesting consequence. If Γ is as in theorem 4, and we consider the set $\Gamma_{sat}= \bigcup_{i=0}^{\infty} \Gamma + (i, 0)$, then Γ_{sat} is dense and connected in the strip, $\widetilde{f}(\Gamma_{sat}) = \Gamma_{sat}$ and, in a sense, all points in Γ_{sat} converge to the left end of $\widetilde{A}$ through iterations of $\tilde{f}$ with a strictly negative velocity. Therefore Γ_{sat} can be seen as part of a dense "unstable manifold of the point L in the L, R-compactification (left and right compactification) of the strip".

Our strategy of proof is the following: Let $f \in Hom_{+}^{trans}(A)$ be such that $\rho(\widetilde{f}\mid_{\mathbb{R}\times\{i\}}) > 0$ for $i = 0, 1$. It is not very difficult to prove that there exists a closed set $B^- \subset \left(\bigcap_{n\leq 0} \widetilde{f}^n(]-\infty, 0] \times [0, 1])\right) \subset]-\infty, 0] \times [0, 1]$, such that $B^- \cap \{0\} \times [0, 1] \neq \emptyset$, every connected component of B^- is unlimited to the left and $\widetilde{f}(B^-) \subset B^-$. Through similar techniques, we will show both that the ω-limit of B^- is empty and that $p(B^-)$ is dense in the annulus. Theorem 3 is derived by using theorem 2 and simple properties of B^-. The other two theorems are proved using the machinery developed in the proof of theorem 1.

2 Basic tools

In this section we define several sets that will play important roles in the proofs of the main theorems and derive some useful relations between them.

2.1 Preliminaries

Let us introduce the set B^- and show some of its properties. To this purpose, we will sometimes make use of the already mentioned left and right compactification of $\widetilde{A} = \mathbb{R} \times [0,1]$, denoted L, R-compactification, that is, we compactify the infinite strip adding two points, L (left end) and R (right end), getting a closed disk, denoted $\widehat{A}$. Clearly $\widetilde{f}$ induces a homeomorphism $\widehat{f} : \widehat{A} \to \widehat{A}$, such that $\widehat{f}(L) = L$ and $\widehat{f}(R) = R$, see figure 1.

Given a real number a, let

$$V_a = \{a\} \times [0,1],$$

$$V_a^- =]-\infty, a] \times [0,1] \text{ and } V_a^+ = [a, +\infty[\times [0,1].$$

Denote the corresponding sets on $\widehat{A}$ by $\widehat{V}_a$, $\widehat{V}_a^-$ and $\widehat{V}_a^+$. We will also denote the sets V_0, V_0^- and V_0^+ simply by V, V^- and V^+ respectively.

If we consider the closed set,

$$\widehat{B} = \bigcap_{n \leq 0} \widehat{f}^n(\widehat{V}^-),$$

we get that, $\widehat{f}(\widehat{B}) \subset \widehat{B}$ and $L \in \widehat{B}$. Denote by $\widehat{B}^-$ the connected component of $\widehat{B}$ which contains L, and by B^- the corresponding set on the strip.

Lemma 1 *: Let $f : A \to A$ be an orientation and boundary components preserving homeomorphism, and let $\tilde{f} : \widetilde{A} \to \widetilde{A}$ be a fixed lift of f. Suppose that for every $a \in \mathbb{R}$, there is a positive integer n such that $\tilde{f}^n(V) \cap V_a \neq \emptyset$ and that $\tilde{f}^i(V_a) \cap V_a \neq \emptyset$ for every integer i. Then $\widehat{B}^- \cap \widehat{V} \neq \emptyset$ (equivalently for the strip: $B^- \cap V \neq \emptyset$)*

Proof:

The proof of this result in a different context appears in Le Calvez [6].

Given $N > 0$, choose a sufficiently small $a < 0$ such that

$$n = \inf\{i > 0 : \widetilde{f}^{-i}(V_a) \cap V \neq \emptyset\} > N.$$

The above is true because as $|a|$ becomes larger, it takes more time for an iterate of V to hit V_a.

From the definition of n we get that: $\widetilde{f}^{-i}(V_a) \subset V^-$, for $i = 0, 1, ..., n-1$ and $\widetilde{f}^{-n}(V_a) \cap V \neq \emptyset$. This implies that there exists a simple continuous arc $\Gamma_N \subset \widetilde{f}^{-n}(V_a^-) \cap V^-$, such that $\widehat{\Gamma}_N$ connects L to $\widehat{V}$ (one endpoint of $\widehat{\Gamma}_N$ is L and the other is in $\widehat{V}$), see figure 2. For this arc, if $1 \leq i \leq n$, we get: $\widetilde{f}^i(\Gamma_N) \subset \widetilde{f}^{-n+i}(V_a^-) \subset V^-$. So,

$$\widehat{\Gamma}_N \subset \bigcap_{i=0}^{n} \widehat{f}^{-i}(\widehat{V}^-),$$

which implies, by taking the limit $N \to \infty \Rightarrow n \to \infty$, that $\widehat{\Gamma}_N$ has a convergent subsequence in the Hausdorff topology to a compact connected set $\widehat{\Gamma} \subset \widehat{A}$, which

connects L to $\widehat{V}$. From its choice, it is clear that $\widehat{\Gamma} \subset \widehat{B}^-$ and thus the lemma is proved. $\square$

Now we:

Claim: If $\tilde{f}$ is transitive then the hypotheses of the previous lemma are satisfied.

Proof:
The transitivity of $\widetilde{f}$ implies that we just have to prove that for every $a \in \mathbb{R}$, $\tilde{f}^i(V_a) \cap V_a \neq \emptyset$ for all integers i. By contradiction, suppose that for some $a \in \mathbb{R}$ and some integer i_0, $\tilde{f}^{i_0}(V_a) \cap V_a = \emptyset$. Without loss of generality, we can suppose that $\tilde{f}^{i_0}(V_a) \subset V_a^-$. Consider the open set

$$W = \bigcup_{j=0}^{i_0-1} \tilde{f}^j(interior(V_a^-)).$$

Clearly, W is open, connected, limited to the right and $\widetilde{f}(W) \subset W$. And this contradicts the existence of a dense orbit. $\square$

Moreover, lemma 1 is true for any rotationless homeomorphism of the annulus with rotation interval not reduced to zero.

Proposition 1 *: If $(f, \widetilde{f})$ is a rotationless homeomorphism such that given $M > 0$, there exists an integer $n > 0$ and a point $\widetilde{z} \in [0,1] \times [0,1]$ such that $\left| p_1(\widetilde{f}^n(\widetilde{z})) \right| > M$, then $\widehat{B}^- \cap \widehat{V} \neq \emptyset$.*

Proof:
From lemma 1 it suffices to prove that for every real number a there is a positive n such that $\widetilde{f}^n(V) \cap V_a \neq \emptyset$, because since $(f, \widetilde{f})$ is rotationless, for all real a, $\widetilde{f}^i(V_a) \cap V_a \neq \emptyset$, for all integers i.

Suppose by contradiction that for some real b, $\widetilde{f}^{-i}(V_b) \cap V = \emptyset$ for all integers $i > 0$. As we said above, $\rho(Leb) = 0$ implies that $\widetilde{f}^l(V_b) \cap V_b \neq \emptyset$ for all integers l, so, if we suppose that $b < 0$, then $\widetilde{f}^{-i}(V_b) \subset V^-$ for all $i > 0$. And this implies that

$$\bigcup_{n \geq 0} \widetilde{f}^{-n}(int(V_b^-)) \subset V^-.$$

As $\widetilde{f}^{-1}\left(\bigcup_{n \geq 0} \widetilde{f}^{-n}(int(V_b^-))\right) \subset \bigcup_{n \geq 0} \widetilde{f}^{-n}(int(V_b^-))$, there is a boundary component of the open connected set $\bigcup_{n \geq 0} \widetilde{f}^{-n}(int(V_b^-))$, denoted K, which is compact connected and intersects $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. Clearly, $\widetilde{f}^{-1}(K) \subset K \cup \left(\bigcup_{n \geq 0} \widetilde{f}^{-n}(V_b^-)\right)$. From $\rho(Leb) = 0$, we get that

$$\widetilde{f}^{-1}(K) \subset K \Rightarrow \widetilde{f}^{-1}(K + (s, 0)) \subset K + (s, 0)$$

for all integers s, something that contradicts the proposition hypotheses. If $b > 0$, an analogous argument using $\widetilde{f}^{-i}(V_b) \subset V^+$ for all $i > 0$ works. $\square$

In the rest of the paper we assume that $f \in Hom_{+}^{trans}(A)$ and $\rho(\widetilde{f}\mid_{\mathbb{R}\times\{i\}}) > 0$ for $i = 0, 1$. So, from lemma 1, we know that $B^- \subset \widetilde{A}$ is a closed set, limited to the right ($B^- \subset V^-$), whose connected components (which may be unique) are all unlimited to the left, and at least one connected component of B^- intersects V.

An important point here is that, as the rotation numbers in the boundaries of the annulus are both positive, B and thus B^-, do not intersect $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ (because $\widetilde{f}(B) \subset B \subset V^-$). So the only part of theorem 1 that still has to be proved is: $p(B^-)$ is dense in the annulus.

2.2 The limit set of B^-

In this subsection we examine some properties of the set

$$\omega(\widehat{B}^-) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{i=n}^{\infty} \widehat{f}^i(\widehat{B}^-)}, \tag{2}$$

a subset of $\widehat{A}$, and the corresponding set $\omega(B^-) \subset \widetilde{A}$.

Since $\widehat{f}(\widehat{B}^-) \subset \widehat{B}^-$, and since $\widehat{B}^-$ is closed, we have

$$\omega(\widehat{B}^-) = \bigcap_{n=0}^{\infty} \widehat{f}^n(\widehat{B}^-),$$

therefore $\omega(\widehat{B}^-)$ is the intersection of a nested sequence of compact connected sets, and so it is also a compact connected set. Moreover, definition (2) implies the following lemma:

Lemma 2 *: $\omega(B^-)$ is a closed, $\widetilde{f}$-invariant set, whose connected components are all unbounded.*

Proof:

Since $L \in \widehat{B}^-$ and $\widehat{f}(L) = L$, we get that $L \in \omega(\widehat{B}^-)$. This implies, since $\omega(\widehat{B}^-)$ is connected, that each connected component of $\omega(B^-)$ is unbounded. The other properties follow directly from the previous considerations. $\square$

Of course, since B^- is closed, we also have that $\omega(B^-) \subset B^-$, and as such, $\omega(B^-) \cap \mathbb{R} \times \{i\} = \emptyset$, $i \in \{0, 1\}$, and $\omega(B^-) \subset V^-$. It is still possible that $\omega(B^-) = \emptyset$, and this is in fact true as we show later, but for the moment we can make use of the fact that both B^- and $\omega(B^-)$ have similar properties to shorten our proofs. For this, let $D \subset \widetilde{A}$ be a non-empty closed set with the following properties:

- $\widetilde{f}(D) \subset D$;
- $D \subset V^-$;
- Every connected component of D is unbounded;

- $D \cap \mathbb{R} \times \{i\} = \emptyset$, $i \in \{0,1\}$;
- If $\widetilde{z} \in D$ then $\widetilde{z} - (1,0) \in D$.

It is easily verified that B^- has these properties, as does $\omega(B^-)$ if it is nonempty, so every result shown for D must hold in the particular cases of interest for us. Later, in the proof of theorem 4, we find another set with the properties listed above.

2.3 On the structure of $p(D) \subset A$

First, let us start with the following lemma:

Lemma 3 *: $\overline{p(D)} \supset S^1 \times \{0\}$, or $\overline{p(D)} \supset S^1 \times \{1\}$.*

Proof:
Suppose that lemma is false. Then, there are points $P_0 \in S^1 \times \{0\}$ and $P_1 \in S^1 \times \{1\}$ such that $\{P_0, P_1\} \cap \overline{p(D)} = \emptyset$. As $\left(\overline{p(D)}\right)^c$ is an open set, there exists $\epsilon > 0$ such that $B_\epsilon(P_i) \cap \overline{p(D)} = \emptyset$, for $i = 0, 1$. As

$$\widetilde{f}(D) \subset D \text{ and } p \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) = f \circ p(\widetilde{x}, \widetilde{y})$$

we get that

$$f(p(D)) \subset p(D) \ \Rightarrow \ f(\overline{p(D)}) \subset \overline{p(D)}.$$

Since $\widetilde{f}$ is transitive, it follows that f is also transitive and so there exists $N > 0$ such that $f^{-N}(B_\epsilon(P_0)) \cap B_\epsilon(P_1) \neq \emptyset$.
Now, we must have that

$$f^{-N}(B_\epsilon(P_0)) \cap \overline{p(D)} = \emptyset,$$

for if this was not true, it would imply $B_\epsilon(P_0) \cap f^N(\overline{p(D)}) \neq \emptyset$, which, in turn, would imply $B_\epsilon(P_0) \cap \overline{p(D)} \neq \emptyset$, because $f^N(\overline{p(D)}) \subset \overline{p(D)}$, contradicting the choice of $\epsilon > 0$.

As $f^{-N}(B_\epsilon(P_0)) \cup B_\epsilon(P_1)$ is disjoint from $\overline{p(D)}$, this implies that there exists a simple continuous arc $\gamma \subset \left(f^{-N}(B_\epsilon(P_0)) \cup B_\epsilon(P_1)\right)$, disjoint from $\overline{p(D)}$, such that its endpoints are in $S^1 \times \{0\}$ and in $S^1 \times \{1\}$. So, if $\widetilde{\gamma}$ is a connected component of $p^{-1}(\gamma)$, then $\widetilde{\gamma} - (i,0) \cap D = \emptyset$ for all integers $i > 0$. And this contradicts the fact that the connected components of D are unlimited to the left. $\square$

Lemma 4 *If $\overline{p(D)} \neq A$, then $\overline{p(D)}^c$ has a single connected component which is dense in A. Moreover, $\overline{p(D)}^c$ contains a homotopically non trivial simple closed curve in the open annulus $S^1 \times]0,1[$.*

Proof:

As $f(\overline{p(D)}) \subset \overline{p(D)}$, we get that $f(\left(\overline{p(D)}\right)^c) \supset \left(\overline{p(D)}\right)^c$, which implies that $f^{-1}(\left(\overline{p(D)}\right)^c) \subset \left(\overline{p(D)}\right)^c$. If $\left(\overline{p(D)}\right)^c$ is not dense, then as f is transitive, there exists an open ball $U \subset A$, $U \cap \left(\overline{p(D)}\right)^c = \emptyset$ and a point $z \in U$ and an integer $n > 0$ such that $f^n(z) \in \left(\overline{p(D)}\right)^c$. And this contradicts the fact that $f^{-1}(\left(\overline{p(D)}\right)^c) \subset \left(\overline{p(D)}\right)^c$, so $\left(\overline{p(D)}\right)^c$ is dense.

Let E be a connected component of $\left(\overline{p(D)}\right)^c$. Assume by contradiction that there is no simple closed curve $\gamma \subset E$ which is homotopically non trivial as a curve of the annulus.

In this case, $p^{-1}(E)$ is not connected and there exists an open connected set $E_{lift} \subset \widetilde{A}$, such that $E_{lift} \cap E_{lift} + (i,0) = \emptyset$, for all integers $i \neq 0$ and

$$p^{-1}(E) = \bigcup_{i=-\infty}^{+\infty} (E_{lift} + (i,0)).$$

As f is transitive and $f^{-1}(\left(\overline{p(D)}\right)^c) \subset \left(\overline{p(D)}\right)^c$, there exists a first $N > 0$, such that $f^{-N}(E) \subset E$ (in other words, for $i \in \{1, 2, ..., N-1\}$, $f^{-i}(E) \cap E = \emptyset$). This means that

$$\begin{array}{c} \widetilde{f}^{-i}(E_{lift}) \cap p^{-1}(E) = \emptyset \text{ for all } i > 0 \text{ which is not a} \\ \text{multiple of } N \text{ and } \widetilde{f}^{-N}(E_{lift}) \subset E_{lift} + (i_0, 0), \\ \text{for some fixed integer } i_0, \text{ which implies that} \\ \widetilde{f}^{-k.N}(E_{lift}) \subset E_{lift} + (k.i_0, 0), \text{ for all integers } k > 0. \end{array} \tag{3}$$

Suppose $i_0 \geq 0$. As $\widetilde{f}$ has a dense orbit, there exists a point $\widetilde{z} \in E_{lift} - (1,0)$ such that $\widetilde{f}^l(\widetilde{z}) \in E_{lift}$, for some $l > 0$. But this means that, $\widetilde{f}^{-l}(E_{lift}) \cap E_{lift} - (1,0) \neq \emptyset$, something that contradicts (3), because we assumed that $i_0 \geq 0$. A similar argument implies that i_0 can not be smaller then zero. Therefore, all connected components of $\left(\overline{p(D)}\right)^c$ contain a homotopically non trivial simple closed curve.

Let E be a connected component of $\left(\overline{p(D)}\right)^c$ and $\gamma_E \subset E$ be a homotopically non trivial simple closed curve. Since f is transitive, $f^{-1}(\gamma_E) \cap \gamma_E \neq \emptyset$. Thus $f^{-1}(E) \cap E \neq \emptyset$ and so, since $f^{-1}(\left(\overline{p(D)}\right)^c) \subset \left(\overline{p(D)}\right)^c$, $f^{-1}(E) \subset E$. But E is open and f is transitive, so E is dense and therefore it is the only connected component of $\left(\overline{p(D)}\right)^c$, proving the lemma. $\square$

So, let

$$\gamma_E \subset \left(\overline{p(D)}\right)^c \cap interior(A) \tag{4}$$

be a homotopically non trivial simple closed curve and let $\gamma_E^- \supset S^1 \times \{0\}$ and $\gamma_E^+ \supset S^1 \times \{1\}$ be the open connected components of γ_E^c. As $\overline{p(D)} \cap \gamma_E = \emptyset$, we obtain that $\overline{p(D)} \subset \gamma_E^- \cup \gamma_E^+$.

Lemma 5 *: Let Γ be a connected component of D. If $\overline{p(D)} \neq A$, we have:*

$$\begin{cases} \textit{if } \overline{p(\Gamma)} \subset \gamma_E^-, \textit{ then } \overline{p(\Gamma)} \supset S^1 \times \{0\} \\ \textit{if } \overline{p(\Gamma)} \subset \gamma_E^+, \textit{ then } \overline{p(\Gamma)} \supset S^1 \times \{1\} \end{cases}$$

Proof:

First note that $\overline{p(\Gamma)} \subset \overline{p(D)}$, which implies that $\overline{p(\Gamma)}^c \supset \overline{p(D)}^c = E$. As E is open, connected and dense, and since $\overline{p(\Gamma)}^c$ is also open, every connected component of $\overline{p(\Gamma)}^c$ contains a point of E. Therefore $\overline{p(\Gamma)}^c$ is also an open connected dense subset of the annulus. Without loss of generality, suppose that $\overline{p(\Gamma)} \subset \gamma_E^-$. This implies that $\overline{p(\Gamma)} \cap S^1 \times \{1\} = \emptyset$. If $\overline{p(\Gamma)}$ does not contain $S^1 \times \{0\}$, then there exists a simple continuous arc λ in the annulus, which avoids $\overline{p(\Gamma)}$ and connects some point $P_0 \in \left(S^1 \times \{0\}\right) \setminus \overline{p(\Gamma)}$ to some point $P_1 \in S^1 \times \{1\}$. This is true because $P_0, P_1 \in \overline{p(\Gamma)}^c$, which is an open connected set. But this means that

$$p^{-1}(\lambda) \cap \Gamma = \emptyset,$$

and this is a contradiction because each connected component of $p^{-1}(\lambda)$ is compact and Γ is connected and unlimited to the left. So, $\overline{p(\Gamma)} \supset S^1 \times \{0\}$. The other possibility ($\overline{p(\Gamma)} \subset \gamma_E^+$) is held in a similar way. □

Without loss of generality we can suppose that there exists a connected component Γ of D that satisfies: $p(\Gamma) \subset \gamma_E^-$. Thus, lemma 5 implies the following fact, which is the most important information of this subsection:

Fact 1 *: If $\overline{p(D)} \neq A$, then there exists a connected component Γ of D that satisfies: $p(\Gamma) \subset \gamma_E^-$, $\overline{p(\Gamma)} \supset S^1 \times \{0\}$, and thus, given $\epsilon > 0$, $p(\Gamma) \cap S^1 \times [0, \epsilon] \neq \emptyset$.*

Proof:

Immediate. □

2.4 On the structure of $D \subset \widetilde{A}$

Let Γ be a connected component of D. We recall that, by the definition of D, Γ is unlimited to the left.

The next proposition is used in several arguments in the remainder of the paper.

Proposition 2 *: Γ^c has only one connected component, which is unlimited.*

Proof:

Clearly, there is one connected component of Γ^c which contains $int(V^+)$, $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$.

So, if by contradiction, we suppose that Γ^c has another connected component, denoted C, contained in V^-, its boundary must be contained in Γ. As $\widetilde{f}^n(\Gamma) \subset V^-$ for all $n \geq 0$, we get that $\widetilde{f}^n(C) \subset V^-$ for all $n \geq 0$. So for every

$\widetilde{z} \in C$, $\limsup_{n\to\infty} p_1 \circ \widetilde{f}^n(\widetilde{z}) < 0$. As C is an open subset of $\widetilde{A}$, this contradicts the transitivity of $\widetilde{f}$. $\Box$

For a connected component Γ of D, let us define

$$m_\Gamma = \sup\{\widetilde{x} \in \mathbb{R} : (\widetilde{x}, \widetilde{y}) \in \Gamma, \text{ for some } \widetilde{y} \in [0,1]\} \leq 0. \tag{5}$$

Consider the connected closed set $\Gamma \cup \{m_\Gamma\} \times [0,1]$. Its complement has two open unbounded connected components in

$$]-\infty, m_\Gamma[\times[0,1],$$

one of which contains $]-\infty, m_\Gamma[\times\{0\}$ (denoted Γ_{down}) and another one which contains $]-\infty, m_\Gamma[\times\{1\}$ (denoted Γ_{up}). It is possible that $(\Gamma \cup \{m_\Gamma\} \times [0,1])^c$ has other unbounded connected components. But only Γ_{up} and Γ_{down} will be of interest to us, because of the following fact, whose proof is an exercise which depends only on the connectivity of Γ (see lemma 7 for a generalization of this result):

Fact 2 *: Given a connected component Γ of D, if Θ is a closed connected set, unlimited to the left, which satisfies $\Theta \cap \Gamma = \emptyset$ and $\Theta \subset\,]-\infty, m_\Gamma[\times[0,1]$, then $\Theta \subset \Gamma_{up}$ or $\Theta \subset \Gamma_{down}$.*

In the following, we will generalize the above construction and present some simple results on the connected components of D. These results will permit us to define an order $\prec$ on the connected components of D. Moreover, it will be clear that any two disjoint closed unlimited connected sets $\Theta_1, \Theta_2 \subset V^-$, which have connected complements will be related by this order, that is either $\Theta_1 \prec \Theta_2$ or $\Theta_2 \prec \Theta_1$. This will be of importance to us, because, if Γ_1, Γ_2 are connected components of D, then $\widetilde{f}(\Gamma_1)$ and $\widetilde{f}(\Gamma_2)$ may not be, they are just contained in connected components of D. As will be explained below, it is possible that a single connected component of D, denoted Θ, contains $\widetilde{f}(\Gamma_1)$ and $\widetilde{f}(\Gamma_2)$ even when $\Gamma_1 \cap \Gamma_2 = \emptyset$ (remember that as Γ_1, Γ_2 are connected components of D, either $\Gamma_1 = \Gamma_2$ or $\Gamma_1 \cap \Gamma_2 = \emptyset$). But in this case, if $\Gamma_1 \prec \Gamma_2$ ($\Gamma_2 \prec \Gamma_1$), as $\widetilde{f}(\Gamma_1)$ and $\widetilde{f}(\Gamma_2)$ are disjoint closed unlimited connected sets which have connected complements, we will show that $\widetilde{f}(\Gamma_1) \prec \widetilde{f}(\Gamma_2)$ ($\widetilde{f}(\Gamma_2) \prec \widetilde{f}(\Gamma_1)$), that is $\widetilde{f}$ preserves the order.

To begin, let $\Gamma \subset D$ be a connected component and let $a \leq 0$ be such that V_a intersects Γ.

Consider the following open set,

$$\Gamma^{comp,a} = \Gamma^c \cap]-\infty, a[\times[0,1]. \tag{6}$$

Lemma 6 *: $\Gamma^{comp,a}$ has at least two (open) connected components, one denoted $\Gamma'_{a,down}$ which contains $]-\infty, a[\times\{0\}$ and one denoted $\Gamma'_{a,up}$ which contains $]-\infty, a[\times\{1\}$.*

Proof:

Suppose the lemma is false. Then, there exists $P \in]-\infty, a[\times\{0\}$, $Q \in]-\infty, a[\times\{1\}$ and a simple continuous arc $\eta \subset \Gamma^{comp,a}$, whose endpoints are P, Q. Clearly, $\eta \subset]-\infty, a[\times[0,1]$. As $\eta \cap \Gamma = \emptyset$, Γ is unlimited to the left and Γ intersects V_a, we obtain that Γ intersects both connected components of η^c, something that contradicts the connectivity of Γ. $\Box$

The arguments contained in the proof of the next proposition will be used many times in the rest of the paper.

Proposition 3 *: Let* $\Gamma \subset D$ *be a connected component and let* $a \leq 0$ *be such that* V_a *intersects* Γ*. Then,* $\Gamma \cap V_a^-$ *has at least one unlimited connected component, which intersects* V_a*.*

Proof:

Let us consider the L, R-compactification of $\widetilde{A} = \mathbb{R} \times [0,1]$, denoted $\widehat{A}$. For every object (point, set, etc) in $\widetilde{A}$, we denote the corresponding object in $\widehat{A}$ by putting a $\widehat{}$ on it.

Let $z_n \in \Gamma \cap V_a^-$ be a sequence such that $p_1(z_n) \overset{n\to\infty}{\to} -\infty$, or equivalently, $\widehat{A} \ni \widehat{z}_n \overset{n\to\infty}{\to} L$.

Also, note that $\widehat{\Gamma}$ is connected, it intersects $\widehat{V}_a$ and contains L. Each $\widehat{z}_n$ belongs to a connected component of $\widehat{\Gamma} \cap \widehat{V}_a^-$, denoted $\widehat{\Gamma}_n$. The connectivity of $\widehat{\Gamma}$ implies that each $\widehat{\Gamma}_n$ intersects $\widehat{V}_a$. Let $\widehat{\Gamma}_{n_i}$ be a convergent subsequence in the Hausdorff topology, $\widehat{\Gamma}_{n_i} \overset{n\to\infty}{\to} \widehat{\Gamma}^*$. This means that, given any open neighborhood $\widehat{N}$ of $\widehat{\Gamma}^*$, for all sufficiently large i, $\widehat{\Gamma}_{n_i}$ is contained in $\widehat{N}$. So $\widehat{\Gamma}^*$ must contain L and must intersect $\widehat{V}_a$. Suppose that $\widehat{\Gamma}^*$ is not contained in $\widehat{\Gamma}$. This means that there exists $\widehat{P} \in \widehat{\Gamma}^*$, with $\widehat{P} \notin \widehat{\Gamma}$. As $\widehat{\Gamma}$ is closed, for some $\epsilon_0 > 0$, $B_{\epsilon_0}(\widehat{P}) \cap \widehat{\Gamma} = \emptyset$, where $B_{\epsilon_0}(\widehat{P}) = \{\widehat{z} \in \widehat{A} : d_{Euclidean}(\widehat{z}, \widehat{P}) < \epsilon_0\}$ and $d_{Euclidean}(\bullet, \bullet)$ is the usual Euclidean distance in $\widehat{A}$. But as $\widehat{\Gamma}_{n_i} \overset{n\to\infty}{\to} \widehat{\Gamma}^*$ in the Hausdorff topology, for all sufficiently large i, $\widehat{\Gamma}^* \subset (\epsilon_0/2 - \textit{neighborhood of } \widehat{\Gamma}_{n_i})$. Thus we get that $d_{Euclidean}(\widehat{P}, \widehat{\Gamma}) \leq d_{Euclidean}(\widehat{P}, \widehat{\Gamma}_{n_i}) < \epsilon_0/2$, something that contradicts the choice of $\widehat{P} \in \widehat{\Gamma}^*$. So $\widehat{\Gamma}^* \subset \widehat{\Gamma}$ and the proposition is proved because although Γ^* may not be connected, it must contain an unlimited connected component which intersects V_a. $\Box$

Before going on, let us define the sets $\Gamma_{a,down}$ and $\Gamma_{a,up}$ as follows:

$$\begin{gathered}\Gamma_{a,down}(\Gamma_{a,up}) = \Gamma'_{a,down}(\Gamma'_{a,up}) \text{ plus all points} \\ \text{in the boundary of } \Gamma'_{a,down}(\Gamma'_{a,up}) \text{ of the form } (a, \widetilde{y})\end{gathered}$$

If Γ is a connected component of D which intersects some vertical V_a, it is possible that $\Gamma \cap V_a^-$ has more then one unlimited connected component. We denote by

$$\left[\Gamma \cap V_a^-\right] = \text{union of all unbounded connected components of } \Gamma \cap V_a^-. \quad (7)$$

Proposition 4 *: Let* $a, b \in \mathbb{R}$ *be such that* $b < a$ *and let* Γ *be a connected component of* D*, which intersects* V_a*. Then* $\Gamma_{b,down} \subset \Gamma_{a,down}$ *and* $\Gamma_{b,up} \subset \Gamma_{a,up}$*.*

Proof:

Let $z \in \Gamma_{b,down}$. This means that there exists a simple continuous arc θ which connects z to a point $z_0 \in]-\infty, b[\times\{0\}$, $\theta \cap \Gamma = \emptyset$ and $\theta \subset \Gamma_{b,down} \subset]-\infty, b[\times[0,1]$. As $a > b$, $\theta \cap \partial\Gamma_{a,down} \subset \theta \cap \Gamma = \emptyset$. As $z_0 \in \Gamma_{a,down}$, we get that $\theta \subset \Gamma_{a,down}$, which implies that $\Gamma_{b,down} \subset \Gamma_{a,down}$. The other inclusion is proved in a similar way. □

Let Γ_1, Γ_2 be two different connected components of D and let V_a be a vertical which intersects Γ_1.

Lemma 7 *: One and only one of the following possibilities must hold:*
$[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$ *or* $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$.

Proof:

First we prove that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down} \cup \Gamma_{1a,up}$. Suppose this is not the case. Then, there exists an unlimited connected component of $\Gamma_2 \cap V_a^-$, denoted Γ_2^*, contained in a connected component of $\Gamma_1^{comp,a}$ (see (6)) different from $\Gamma_{1a,down}$ and $\Gamma_{1a,up}$. Denote this component by $\Gamma_{1a,mid}$. Fix some $P \in \Gamma_2^*$. As $P \notin \Gamma_1$, there exists $\epsilon > 0$ such that $B_\epsilon(P) \cap \Gamma_1 = \emptyset$. Now, let $\alpha' \subset \mathbb{R}\times]0,1[$ be a simple continuous arc which connects P to $(1, 0.5)$, totally contained in Γ_1^c, which is an open connected set that contains P and $(1, 0.5)$. Moreover, as $]0,+\infty[\times[0,1] \subset \Gamma_1^c$, we can take α' so that it does not intersect $]1,+\infty[\times\{0.5\}$. Finally, let α be a simple continuous arc given by $[1,+\infty[\times\{0.5\}$ plus a continuous part of α', whose endpoints are $(1, 0.5)$ and some point in Γ_2^*, so that $\alpha \cap \Gamma_2^*$ consists of only its end point (clearly, this end point may not be P).

Properties of $\alpha \cup \Gamma_2^*$:

- $\alpha \cup \Gamma_2^*$ is a closed, connected set, disjoint from $\mathbb{R} \times \{0,1\}$;
- $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are in different connected components of $(\alpha \cup \Gamma_2^*)^c$;
- α is limited to the left, that is, there exists a number $M > 0$ such that, for all points $\widetilde{z}$ in α, $p_1(\widetilde{z}) > -M$;
- $(\alpha \cup \Gamma_2^*) \cap \Gamma_1 = \emptyset$;

Let us choose $b < a$ such that $\alpha \subset V_{b+1/2}^+$. By proposition 4, $\Gamma_{1b,down} \subset \Gamma_{1a,down}$ and $\Gamma_{1b,up} \subset \Gamma_{1a,up}$, so we get that $\Gamma_2^* \cap (\Gamma_{1b,down} \cup \Gamma_{1b,up}) = \emptyset$. Now let $\beta_0 \subset \Gamma_{1b,down}$ and $\beta_1 \subset \Gamma_{1b,up}$ be simple continuous arcs which satisfy the following:

- β_0 connects a point of $]-\infty, b[\times\{0\}$ to a point of Γ_1;
- β_1 connects a point of $]-\infty, b[\times\{1\}$ to a point of Γ_1;

So the following conditions hold

$$(\beta_0 \cup \beta_1) \cap \Gamma_2^* = \emptyset \text{ and } (\beta_0 \cup \beta_1) \subset V_b^- \Rightarrow (\beta_0 \cup \beta_1) \cap \alpha = \emptyset$$

and thus

$$(\beta_0 \cup \Gamma_1 \cup \beta_1) \cap (\alpha \cup \Gamma_2^*) = \emptyset,$$

something that contradicts the fact that $(\beta_0 \cup \Gamma_1 \cup \beta_1)$ is a closed connected set and the "**Properties of** $\alpha \cup \Gamma_2^*$" listed above, see figure 3. So, $[\Gamma_2 \cap V_a^-] \subset (\Gamma_{1a,down} \cup \Gamma_{1a,up})$.

Suppose now that for some $\Gamma_2^*, \Gamma_2^{**} \in [\Gamma_2 \cap V_a^-]$, we have $\Gamma_2^* \subset \Gamma_{1a,down}$ and $\Gamma_2^{**} \subset \Gamma_{1a,up}$. In the same way as above, there exists a simple continuous arc $\alpha \subset \mathbb{R} \times]0,1[$ which contains $[1,+\infty[\times \{0.5\}$ and connects some point of Γ_1 to $(1, 0.5)$, in a way that $\alpha \subset \Gamma_2^c$ and α intersects Γ_1 only at its end point. Clearly, $(\alpha \cup \Gamma_1)$ is a closed connected set, which satisfies: $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are in different connected components of $(\alpha \cup \Gamma_1)^c$.

Again, as above, let us choose $b < a$ such that $\alpha \subset V_{b+1}^+$. Proposition 3 implies that $\left[\Gamma_2^* \cap V_b^-\right]$ and $\left[\Gamma_2^{**} \cap V_b^-\right]$ are non-empty. From what we did above, we get that $\left[\Gamma_2^* \cap V_b^-\right] \cup \left[\Gamma_2^{**} \cap V_b^-\right] \subset \Gamma_{1b,down} \cup \Gamma_{1b,up}$.

If, $\left[\Gamma_2^* \cap V_b^-\right] \cap \Gamma_{1b,up} \neq \emptyset \Rightarrow \Gamma_2^* \cap \Gamma_{1b,up} \neq \emptyset$, which implies, by proposition 4, that $\Gamma_2^* \cap \Gamma_{1a,up} \neq \emptyset$, a contradiction. So, $\left[\Gamma_2^* \cap V_b^-\right] \cap \Gamma_{1b,up} = \emptyset$ and a similar argument gives $\left[\Gamma_2^{**} \cap V_b^-\right] \cap \Gamma_{1b,down} = \emptyset$. So, $\left[\Gamma_2^* \cap V_b^-\right] \subset \Gamma_{1b,down}$ and $\left[\Gamma_2^{**} \cap V_b^-\right] \subset \Gamma_{1b,up}$. Thus, there exists a simple continuous arc β_0 contained in $\Gamma_{1b,down}$ which connects a point of Γ_2^* to some point in $]-\infty, b[\times \{0\}$. Similarly, there exists a simple continuous arc β_1 contained in $\Gamma_{1b,up}$ which connects a point of Γ_2^{**} to some point in $]-\infty, b[\times \{1\}$, see figure 4. But $(\beta_0 \cup \Gamma_2 \cup \beta_1)$ is a closed connected set and by construction of β_0 and β_1,

$$(\beta_0 \cup \Gamma_2 \cup \beta_1) \cap (\alpha \cup \Gamma_1) = \emptyset,$$

which is a contradiction, completing the proof of the lemma. $\square$

The previous results will be used in what follows in order to define a complete ordering among the connected components of D.

Let Γ_1 and Γ_2 be different connected components of D and let $a \in \mathbb{R}$ be such that Γ_1 and Γ_2 intersect V_a. We say that $\Gamma_2 \prec_a \Gamma_1$, if $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$.

Lemma 8 *: Given* Γ_1, Γ_2 *and* $a \in \mathbb{R}$ *as above, either* $\Gamma_2 \prec_a \Gamma_1$ *or* $\Gamma_1 \prec_a \Gamma_2$.

Proof:

From lemma 7, either $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$ or $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$. In the first possibility, $\Gamma_2 \prec_a \Gamma_1$. So we are left to show that, if $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$, then $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$, which means that $\Gamma_1 \prec_a \Gamma_2$.

Thus, let us suppose that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$ and $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,up}$. If we arrive at a contradiction, the lemma will be proved.

The argument here is very similar to the one used in the proof of lemma 7. First, choose a simple continuous arc $\alpha \subset \mathbb{R} \times]0,1[$ which contains $[1,+\infty[\times \{0.5\}$ and connects some point of Γ_1 to $(1, 0.5)$, in a way that $\alpha \subset \Gamma_2^c$ and α intersects Γ_1 only at its end point. Clearly, $(\alpha \cup \Gamma_1)$ is a closed connected set, which satisfies: $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are in different connected components of $(\alpha \cup \Gamma_1)^c$.

As $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$, there exists an element of $[\Gamma_2 \cap V_a^-]$, denoted Γ_2^*, which by definition is closed, connected, unlimited to the left and is contained in $\Gamma_{1a,up}$. Again, let us choose $b < a$ such that $\alpha \subset V_{b+1}^+$.

As in the end of the proof of lemma 7, we get that $\left[\Gamma_2^* \cap V_b^-\right] \subset \Gamma_{1b,up}$. So, there exists a simple continuous arc β_1 contained in $\Gamma_{1b,up}$ which connects a point of Γ_2^* to some point in $]-\infty, b[\times\{1\}$. Clearly, $\beta_1 \cap \Gamma_1 = \emptyset$.

As $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,up}$, an argument similar to the one used to prove proposition 3 implies:

Proposition 5 *: There exists a real number $c \leq b$, such that $(\Gamma_1 \cap V_c^-) \cap \Gamma_{2a,down} = \emptyset$.*

Proof:

Suppose by contradiction, that the fact is not true. Then, there is a sequence of points $z_n \in \Gamma_1 \cap \Gamma_{2a,down}$, such that $p_1(z_n) \overset{n\to\infty}{\to} -\infty$, or equivalently, $\widehat{A} \ni \widehat{z}_n \overset{n\to\infty}{\to} L$. Each $\widehat{z}_n$ belongs to a connected component of $\widehat{\Gamma}_1 \cap \widehat{V}_a^-$, denoted $\widehat{\Gamma}_{1,n} \subset closure\left(\widehat{\Gamma}_{2a,down}\right)$. The connectivity of Γ_1 and the fact that it intersects V_a implies that each $\widehat{\Gamma}_{1,n}$ intersects $\widehat{V}_a$. Let $\widehat{\Gamma}_{1,n_i}$ be a convergent subsequence in the Hausdorff topology, $\widehat{\Gamma}_{1,n_i} \overset{n\to\infty}{\to} \widehat{\Gamma}^*$. This means that, given any open neighborhood $\widehat{N}$ of $\widehat{\Gamma}^*$, for all sufficiently large i, $\widehat{\Gamma}_{1,n_i}$ is contained in $\widehat{N}$. So $\widehat{\Gamma}^*$ must contain L and must intersect $\widehat{V}_a$. Suppose that $\widehat{\Gamma}^*$ is not contained in $\widehat{\Gamma}_1$. This means that there exists $\widehat{P} \in \widehat{\Gamma}^*$, with $\widehat{P} \notin \widehat{\Gamma}_1$. As $\widehat{\Gamma}_1$ is closed, for some $\epsilon_0 > 0$, $B_{\epsilon_0}(\widehat{P}) \cap \widehat{\Gamma}_1 = \emptyset$, where $B_{\epsilon_0}(\widehat{P}) = \{\widehat{z} \in \widehat{A} : d_{Euclidean}(\widehat{z}, \widehat{P}) < \epsilon_0\}$ and $d_{Euclidean}(\bullet, \bullet)$ is the usual Euclidean distance in $\widehat{A}$. But as $\widehat{\Gamma}_{n_i} \overset{n\to\infty}{\to} \widehat{\Gamma}^*$ in the Hausdorff topology, for all sufficiently large i, $\widehat{\Gamma}^* \subset (\epsilon_0/2 - neighborhood\ of\ \widehat{\Gamma}_{n_i})$. Thus we get that $d_{Euclidean}(\widehat{P}, \widehat{\Gamma}_1) \leq d_{Euclidean}(\widehat{P}, \widehat{\Gamma}_{1,n_i}) < \epsilon_0/2$, something that contradicts the choice of $\widehat{P} \in \widehat{\Gamma}^*$. So $\widehat{\Gamma}^* \subset \widehat{\Gamma}_1$. Clearly $\widehat{\Gamma}^* \subset closure\left(\widehat{\Gamma}_{2a,down}\right)$ and, as $\Gamma_1 \cap \Gamma_2 = \emptyset$, we get, by lemma 7 that $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$, a contradiction with our hypothesis. $\Box$

Now let us look at $\Gamma_{2c,down} \subset \Gamma_{2b,down}$, where c comes from proposition 5. Thus, $\Gamma_1 \cap \Gamma_{2c,down} = \emptyset$. So, there exists a simple continuous arc β_0 which connects a point of Γ_2 to some point in $]-\infty, c[\times\{0\}$, in a way that $\beta_0 \cap \Gamma_2$ is one extreme of β_0, denoted m_0, and $\beta_0 \backslash \{m_0\} \subset \Gamma_{2c,down}$, which implies that $\beta_0 \cap \Gamma_1 = \emptyset$ and $\beta_0 \cap \alpha = \emptyset$. So, $(\beta_0 \cup \Gamma_2 \cup \beta_1)$ is a closed connected set, which intersects $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. And, by construction

$$(\beta_0 \cup \Gamma_2 \cup \beta_1) \cap (\alpha \cup \Gamma_1) = \emptyset,$$

a contradiction. So if $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$, then $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$, which implies that $\Gamma_1 \prec_a \Gamma_2$ and the lemma is proved. $\Box$

Finally, in order to present a good definition of order, we need the following lemma:

Lemma 9 *: Let Γ_1 and Γ_2 be different connected components of D and let $a, b \in \mathbb{R}$ be such that Γ_1 and Γ_2 intersect V_a and V_b. Then we have the following:*

$$\begin{array}{lll} \Gamma_1 \prec_a \Gamma_2 & \Leftrightarrow & \Gamma_1 \prec_b \Gamma_2 \\ \Gamma_2 \prec_a \Gamma_1 & \Leftrightarrow & \Gamma_2 \prec_b \Gamma_1 \end{array}$$

Proof:

Suppose that $b < a$ and $\Gamma_2 \prec_a \Gamma_1 \Leftrightarrow [\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$. Proposition 5 tells us that $(\Gamma_2 \cap V_a^-) \cap \Gamma_{1a,up}$ is a limited set. So, as $\Gamma_{1b,up} \subset \Gamma_{1a,up}$, $\left[\Gamma_2 \cap V_b^-\right]$ must be contained in $\Gamma_{1b,down}$, which means that $\Gamma_2 \prec_b \Gamma_1$. The other implications are proved in a similar way. □

So given Γ_1 and Γ_2, two different connected components of D, if $a \in \mathbb{R}$ is such that Γ_1 and Γ_2 intersect V_a, we can define an order $\prec$ between them as explained above and this order is independent of the choice of a. We only need the following condition: V_a must intersect Γ_1 and Γ_2.

Also, let us prove the following associativity lemma:

Lemma 10 *: If $\Gamma_1, \Gamma_2, \Gamma_3$ are connected components of D, such that $\Gamma_1 \prec \Gamma_2$ and $\Gamma_2 \prec \Gamma_3$, then $\Gamma_1 \prec \Gamma_3$.*

Proof:

Let $a \in \mathbb{R}$ be such that Γ_1, Γ_2 and Γ_3 intersect V_a. Then, $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$ and $[\Gamma_2 \cap V_a^-] \subset \Gamma_{3a,down}$. In the proof of lemma 8, we proved that if Θ and Λ are different connected components of D and $a \in \mathbb{R}$ is such that Θ and Λ intersect V_a then, $[\Theta \cap V_a^-] \subset \Lambda_{a,down}$ implies $[\Lambda \cap V_a^-] \subset \Theta_{a,up}$. So, $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$ and $[\Gamma_3 \cap V_a^-] \subset \Gamma_{2a,up}$. Now, using proposition 5, let us choose $b \leq a$ such that the following inclusions hold:

$$\begin{array}{l} \Gamma_3 \cap V_b^- \subset \Gamma_{2b,up} \\ \Gamma_1 \cap V_b^- \subset \Gamma_{2b,down} \\ \Gamma_2 \cap V_b^- \subset \Gamma_{3b,down} \end{array} \tag{8}$$

Finally, let us prove that $\Gamma_{2b,down} \subset \Gamma_{3b,down}$.

If this is not the case, then there exists a simple continuous arc $\alpha \subset \Gamma_{2b,down}$ that connects a point from $]-\infty, b[\times\{0\}$ to a point $P \notin \Gamma_{3b,down}$. Thus α intersects Γ_3, a contradiction with expression (8). So, $\Gamma_1 \cap V_b^- \subset \Gamma_{2b,down} \subset \Gamma_{3b,down}$, which implies that $\left[\Gamma_1 \cap V_b^-\right] \subset \Gamma_{3b,down} \Leftrightarrow \Gamma_1 \prec \Gamma_3$ and the lemma is proved. □

Our next objective is to show that $\widetilde{f}$ preserves the order just defined. First, note that if Γ is a connected component of D, as $\widetilde{f}(D) \subset D$ and $\widetilde{f}(\Gamma)$ is connected, unlimited to the left, there exists a connected component of D, denoted Γ^+, that contains $\widetilde{f}(\Gamma)$.

We have

Lemma 11 *: Let Γ_1, Γ_2 be connected components of B^- and suppose $\Gamma_1 \prec \Gamma_2$. Then, $\widetilde{f}(\Gamma_1) \prec \widetilde{f}(\Gamma_2)$ and so, if $\Gamma_1^+ \neq \Gamma_2^+$, then $\Gamma_1^+ \prec \Gamma_2^+$.*

Proof:

Suppose that $\Gamma_2^+ \prec \Gamma_1^+$. As $\Gamma_1 \prec \Gamma_2$, for any $a \in \mathbb{R}$ such that Γ_1 and Γ_2 intersect V_a, the proof of lemma 8 implies that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$. From proposition 5, there exists a sufficiently small $b < 0$ such that:

$$\begin{array}{l} \Gamma_1 \cap V_b^- \subset \Gamma_{2b,down} \\ \Gamma_2 \cap V_b^- \subset \Gamma_{1b,up} \\ \Gamma_2^+ \cap V_b^- \subset \Gamma_{1b,down}^+ \end{array} \tag{9}$$

Let $c < b$ be such that $\widetilde{f}^{\pm 1}(V_c) \cap V_b = \emptyset$. From our previous results, we get that

$$\begin{array}{l} \Gamma_1 \cap V_c^- \subset \Gamma_{2c,down} \\ \Gamma_2 \cap V_c^- \subset \Gamma_{1c,up} \\ \Gamma_2^+ \cap V_c^- \subset \Gamma_{1c,down}^+ \end{array} .$$

So, there exists a simple continuous arc $\alpha \subset \Gamma_{1c,down}^+ \subset V_c^-$ that connects a point from $]-\infty, c[\times\{0\}$ to a point $P \in \widetilde{f}(\Gamma_2) \subset \Gamma_2^+$. From the choice of c, $\widetilde{f}^{-1}(\alpha) \subset V_b^-$ and it connects a point from $]-\infty, b[\times\{0\}$ to $\widetilde{f}^{-1}(P) \in \Gamma_2$. As $\alpha \subset \Gamma_{1c,down}^+$, $\alpha \cap \Gamma_1^+ = \emptyset$, so $\widetilde{f}^{-1}(\alpha) \cap \Gamma_1 = \emptyset$. Thus, $\widetilde{f}^{-1}(\alpha) \subset \Gamma_{1b,down}$, which implies that $\Gamma_2 \cap \Gamma_{1b,down} \neq \emptyset$ and this contradicts (9). So, either $\Gamma_1^+ = \Gamma_2^+$, or $\Gamma_1^+ \prec \Gamma_2^+$, because what the proof presented above really shows is that

$$\Gamma_1 \prec \Gamma_2 \Rightarrow \widetilde{f}(\Gamma_1) \prec \widetilde{f}(\Gamma_2). \ \Box$$

In the particular cases where $D = B^-$ or when $D = \omega(B^-)$, we can show that $\widetilde{f}^{-1}$ is also an order-preserving transformation. This is more clearly seen when $D = \omega(B^-)$ since $\omega(B^-)$ is $\widetilde{f}$-invariant. If Γ is a connected component of $\omega(B^-)$, so is $\widetilde{f}^{-1}(\Gamma)$, which we call Γ^-. It is then a simple consequence of the previous lemma that, if $\Gamma_1, \Gamma_2 \subset \omega(B^-)$, with $\Gamma_1 \prec \Gamma_2$, then $\Gamma_1^- \prec \Gamma_2^-$.

When $D = B^-$, if Γ is a connected component of B^-, then $\widetilde{f}^{-1}(\Gamma)$ is also closed connected, unlimited to the left, and limited to the right. The only possibility that may prevent $\widetilde{f}^{-1}(\Gamma)$ from being contained in a connected component of B^- is the following: $\widetilde{f}^{-1}(\Gamma) \cap]0, +\infty[\times[0,1] \neq \emptyset$. There are 2 possibilities:

1. $\widetilde{f}^{-1}(\Gamma) \cap]0, +\infty[\times[0,1] = \emptyset$. As $\widetilde{f}\left(\widetilde{f}^{-1}(\Gamma)\right) = \Gamma \subset B^-$, there is a connected component of B^-, denoted Γ^-, which satisfies $\Gamma^- \supset \widetilde{f}^{-1}(\Gamma)$ and $\widetilde{f}(\Gamma^-) \cap \Gamma \neq \emptyset$. This implies that$\Rightarrow$ $\widetilde{f}(\Gamma^-) \subset \Gamma$, and so $\Gamma^- = \widetilde{f}^{-1}(\Gamma)$.

2. $\widetilde{f}^{-1}(\Gamma) \cap]0, +\infty[\times[0,1] \neq \emptyset$. As $\left[\widetilde{f}^{-1}(\Gamma) \cap V^-\right]$ has at least one connected component, denoted Γ^*, we get that $\widetilde{f}(\Gamma^*) \subset \Gamma \subset B^-$, so there is a connected component of B^-, denoted Γ^-, which contains Γ^* and $\widetilde{f}(\Gamma^-) \subset \Gamma$ because $\widetilde{f}(\Gamma^-) \cap \Gamma \neq \emptyset$. Note that in this case, Γ^- may not be unique.

We can still formulate the following result:

Lemma 12 *: Let Γ_1, Γ_2 be connected components of D and suppose $\Gamma_1 \prec \Gamma_2$. Then, for any choice of Γ_1^- and Γ_2^-, we have $\Gamma_1^- \prec \Gamma_2^-$.*

Proof: Using lemma 11, as $\Gamma_1 \neq \Gamma_2$, if $\Gamma_2^- \prec \Gamma_1^-$, then $\Gamma_2 \prec \Gamma_1$, therefore we must have $\Gamma_1^- \prec \Gamma_2^-$. □

We now, for a fixed connected component Γ of D, consider the covering mapping $p\mid_\Gamma$. It may or may not be injective. We examine the consequences in each case:

2.4.1 The covering mapping $p\mid_\Gamma$ is not injective

This means that there exists $\widetilde{z} \in \widetilde{A}$ and an integer $s > 0$ such that $\widetilde{z}, \widetilde{z} + (s,0) \in \Gamma$. So, $\Gamma \cap \Gamma - (s,0) \neq \emptyset$. The last property of D tell us that $\Gamma - (s,0) \subset D$. But this implies that

$$\Gamma - (s,0) \subset \Gamma, \tag{10}$$

because Γ is a connected component of D.

Suppose that $\Gamma - (1,0)$ is not contained in Γ. As $\Gamma - (1,0) \subset D$, we get that $\Gamma - (1,0) \cap \Gamma = \emptyset$. As $\Gamma - (1,0)$ does not intersect $V_{m_\Gamma} = \{m_\Gamma\} \times [0,1]$, lemma 7 implies that either $\Gamma - (1,0) \subset \Gamma_{down}$ or $\Gamma - (1,0) \subset \Gamma_{up}$. Suppose it is contained in Γ_{up}.

Proposition 6 *: If $\Gamma - (1,0) \subset \Gamma_{up}$, then $\Gamma - (i,0) \subset \Gamma_{up}$ for all integers $i > 1$.*

Proof:

Suppose there exists $s_0 > 1$ (the smallest one) such that $\Gamma - (s_0,0) \subset \Gamma$. This means that $\Gamma, \Gamma - (1,0), ..., \Gamma - (s_0 - 1, 0)$ are all disjoint.

As $\Gamma - (1,0) \subset \Gamma_{up}$ (which implies that $\Gamma \prec \Gamma - (1,0)$), we get that $\Gamma - (s,0) \cap \Gamma - (s+1,0) = \emptyset$ and $\Gamma - (s,0) \prec \Gamma - (s+1,0)$, for all integers $s > 0$. So, in particular, using lemma 10, we obtain the following implications:

$$\begin{array}{l} 1)\ \Gamma \prec \Gamma - (1,0) \prec \Gamma - (2,0) \prec \Gamma - (3,0) \prec ... \prec \Gamma - (s_0 - 1, 0) \\ 2)\ \Gamma - (s_0 - 1, 0) \prec \Gamma - (s_0, 0). \end{array} \tag{11}$$

So, as $\Gamma - (s_0,0) \subset \Gamma$ and $\Gamma \cap \Gamma - (s_0 - 1, 0) = \emptyset$, we get from 2) of (11) that $\Gamma - (s_0 - 1, 0) \prec \Gamma$, a contradiction with 1) of (11). Thus for all integers $i > 0$, $\Gamma \cap \Gamma - (i,0) = \emptyset$ and so

$$\Gamma \prec \Gamma - (1,0) \prec \Gamma - (2,0) \prec \Gamma - (3,0) \prec ... \prec \Gamma - (i,0).$$

If $a \in \mathbb{R}$ is such that Γ and $\Gamma - (i,0)$ intersect V_a, then, $[\Gamma \cap V_a^-] \subset \Gamma - (i,0)_{a,down}$ $\Leftrightarrow [\Gamma - (i,0) \cap V_a^-] \subset \Gamma_{a,up}$. As $\Gamma_{a,up} \subset \Gamma_{up}$, we get that $\Gamma - (i,0) \cap \Gamma_{up} \neq \emptyset$, and so $\Gamma - (i,0) \subset \Gamma_{up}$, because $\Gamma - (i,0) \cap V_{m_\Gamma} = \emptyset$. □

Therefore, if the map $p\mid_\Gamma$ is not injective, then

$$\Gamma - (1,0) \subset \Gamma. \tag{12}$$

We will call Γ a non-injective component.

2.4.2 The covering mapping $p\,|_\Gamma$ is injective

This implies that $\Gamma \cap \Gamma + (s,0) = \emptyset$, for all integers $s \neq 0$. In particular, $\Gamma \cap \Gamma - (1,0) = \emptyset$ and we use this relation to describe the asymptotic behavior of $p(\Gamma)$ around the annulus. As we explained just before defining the order $\prec$, any two unlimited closed connected disjoint subsets of $V^- \subset \widetilde{A}$ which have connected complements, denoted Θ_1 and Θ_2, are related by $\prec$, that is, either $\Theta_1 \prec \Theta_2$ or $\Theta_2 \prec \Theta_1$. So, we say that Γ is a down component of D if $\Gamma \prec \Gamma - (1,0)$ and, analogously, Γ is an up component if $\Gamma - (1,0) \prec \Gamma$.

Lemma 13 *: If $\Gamma \subset D$ is a down component, then $dist(\Gamma, \mathbb{R} \times \{1\}) > 0$ and analogously, if $\Gamma \subset D$ is an up component, then $dist(\Gamma, \mathbb{R} \times \{0\}) > 0$.*

Proof:

In both cases, the proof is analogous, so suppose Γ is a down component. This means that $\Gamma - (1,0)$ is contained in Γ_{up}.

Thus, for any $\widetilde{x} < m_\Gamma$ (see(5)), if we consider the segment $\{\widetilde{x}\} \times [0, \widetilde{y}^*]$, where

$$\widetilde{y}^* = \widetilde{y}^*(\widetilde{x}) = \sup\{\widetilde{y} \in]0,1[: \Gamma \cap \{\widetilde{x}\} \times [0,\widetilde{y}] = \emptyset\}, \tag{13}$$

we get that $\Gamma - (1,0) \cap \{\widetilde{x}\} \times [0, \widetilde{y}^*] = \emptyset$.

Now, consider a point $(m_\Gamma - 1, \widetilde{y}_\Gamma) \in \Gamma - (1,0)$ and a simple continuous arc $\gamma \subset int(\widetilde{A})$, such that:

i) $\gamma \cap (\Gamma - (1,0)) = (m_\Gamma - 1, \widetilde{y}_\Gamma)$

ii) $\gamma \cap \Gamma = \emptyset$

iii) the endpoints of γ are $(m_\Gamma - 1, \widetilde{y}_\Gamma)$ and $(m_\Gamma + 1, 0.5)$

iv) $\gamma \cap \{m_\Gamma + 1\} \times [0,1] = (m_\Gamma + 1, 0.5)$

As $\Gamma - (1,0) \subset \Gamma_{up}$ and $(\Gamma \cup \Gamma - (1,0))^c$ is connected, it is possible to choose γ as above, see figure 5.

The complement of the closed connected set $\Gamma - (1,0) \cup \gamma \cup \{m_\Gamma + 1\} \times [0,1]$ has exactly two connected components in $]-\infty, m_\Gamma + 1[\times [0,1]$, one containing $]-\infty, m_\Gamma + 1[\times \{0\}$, denoted $\Gamma - (1,0)_{down}$ and the other containing $]-\infty, m_\Gamma + 1[\times \{1\}$, denoted $\Gamma - (1,0)_{up}$. Note that this construction is not unique, because we may have more then one point in $\Gamma - (1,0) \cap \{m_\Gamma - 1\} \times [0,1]$. Nevertheless, for any such choice, $\Gamma \subset \Gamma - (1,0)_{down}$. This follows from the fact that, for any

$$\widetilde{x} < \min\,[m_\Gamma, \min\{\widetilde{x} \in \mathbb{R} : (\widetilde{x}, \widetilde{y}) \in \gamma, \text{ for some } \widetilde{y} \in]0,1[\}] - 10,$$

the segment $\{\widetilde{x}\} \times [0, \widetilde{y}^*]$ (see (13)) does not intersect $\Gamma - (1,0) \cup \gamma \cup \{m_\Gamma + 1\} \times [0,1]$ and $(\widetilde{x}, \widetilde{y}^*) \in \Gamma$.

Now suppose, by contradiction, that $dist(\Gamma, \mathbb{R} \times \{1\}) = 0$. As Γ is closed and $\Gamma \cap \mathbb{R} \times \{1\} = \emptyset$, we get that for every

$$M \leq M_0 = \min\,[m_\Gamma - 10, \min\{\widetilde{x} \in \mathbb{R} : (\widetilde{x}, \widetilde{y}) \in \gamma, \text{ for some } \widetilde{y} \in]0,1[\}] - 10$$

there exists $\epsilon > 0$ such that if $\widetilde{z} \in \Gamma \cap \mathbb{R} \times [1-\epsilon, 1]$, then $p_1(\widetilde{z}) < M$. So, for M_0 and $\epsilon > 0$ as above, let us choose a point $\widetilde{z}_0 \in \Gamma \cap \mathbb{R} \times [1-\epsilon, 1]$ such that

$p_1(\widetilde{z}_0) \geq p_1(\widetilde{z})$ for all $\widetilde{z} \in \Gamma \cap \mathbb{R}\times[1-\epsilon, 1]$ and

$$dist(\widetilde{z}_0, \mathbb{R}\times\{1\}) < dist(\widetilde{z}, \mathbb{R}\times\{1\}) \text{ for all}$$
$$\widetilde{z} \in \Gamma \cap \{p_1(\widetilde{z}_0)\} \times [1-\epsilon, 1] \text{ with } \widetilde{z} \neq \widetilde{z}_0.$$

Intuitively, if we start going left from $\{m_\Gamma\} \times [0,1]$, $\widetilde{z}_0$ is the point of Γ with largest possible $\widetilde{x}$ and $\widetilde{y}$ coordinates, that belongs to $\mathbb{R}\times[1-\epsilon, 1]$.

Now consider a closed vertical segment l contained in $\mathbb{R}\times[1-\epsilon, 1]$, starting at $\widetilde{z}_0$ and ending at $\mathbb{R}\times\{1\}$. By construction of l, $l \cap \Gamma = \widetilde{z}_0$. As $\Gamma \subset \Gamma - (1,0)_{down}$ and $l \cap (\gamma \cup \{m_\Gamma + 1\} \times [0,1]) = \emptyset$, we get that $l \cap \Gamma - (1,0) \neq \emptyset$. So, there exists $\widetilde{z}_1 \in l \cap \Gamma - (1,0)$ which implies that $\widetilde{z}_1 + (1,0) \in l + (1,0) \cap \Gamma$. And this contradicts the choice of $\widetilde{z}_0$. $\square$

In this case we will say that Γ is an injective component.

3 Proof of theorem 2

Suppose, by contradiction, that $\omega(B^-) \neq \emptyset$. This implies, by lemma 3, that either $S^1 \times \{0\} \subset \overline{p(\omega(B^-))}$ or $S^1 \times \{1\} \subset \overline{p(\omega(B^-))}$. Let us assume, without loss of generality, that $S^1 \times \{0\} \subset \overline{p(\omega(B^-))}$.

Since the rotation number of $\widetilde{f}$ restricted to $S^1 \times \{0\}$ is strictly positive, there exists $\sigma > 0$ such that $p_1(\widetilde{f}(\widetilde{x}, 0)) > \widetilde{x} + 2\sigma$ for all $\widetilde{x} \in \mathbb{R}$. Let $\epsilon > 0$ be sufficiently small such that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times [0, \epsilon]$, $p_1 \circ \tilde{f}(\tilde{x}, \widetilde{y}) > \tilde{x} + \sigma$.

As $S^1 \times \{0\} \subset \overline{p(\omega(B^-))}$, there is a real a such that

$$\omega(B^-) \cap \{a\} \times [0, \epsilon] \neq \emptyset. \tag{14}$$

The fact that $\omega(B^-)$ is closed implies that there must be a $\delta \leq \epsilon$ such that $(a, \delta) \in \omega(B^-)$, and such that for all $0 \leq \tilde{y} < \delta, (a, \tilde{y}) \notin \omega(B^-)$. In other words, (a, δ) is the "lowest" point of $\omega(B^-)$ in $\{a\} \times [0, \epsilon]$. We denote by v the segment $\{a\} \times [0, \delta[$.

Let Θ_1 be the (unbounded) connected component of $\omega(B^-)$ that contains (a, δ). Let Ω be the connected component of $(\Theta_1 \cup v)^c$ that contains $]-\infty, a[\times\{0\}$. Of course, $\partial\Omega \subset \Theta_1 \cup v$, and $\partial\widetilde{f}(\Omega) \subset \widetilde{f}(\Theta_1) \cup \widetilde{f}(v)$.

Note that, since $\omega(B^-) \cap v = \emptyset$, and since $\omega(B^-)$ is $\widetilde{f}$-invariant,

$$\widetilde{f}(\Theta_1) \cap v = \Theta_1 \cap \widetilde{f}(v) = \emptyset.$$

Also, by the choice of $\epsilon > 0, \widetilde{f}(v) \cap v = \emptyset$.

Proposition 7 *: The following inclusion holds:* $\Omega \subset \tilde{f}(\Omega)$

Proof:
There are 2 possibilities:

1. $\tilde{f}(\Theta_1) \neq \Theta_1 \Rightarrow \tilde{f}(\Theta_1) \cap \Theta_1 = \emptyset$
2. $\tilde{f}(\Theta_1) = \Theta_1$

Assume first that $\tilde{f}(\Theta_1) \cap \Theta_1 = \emptyset$. Then

$$\partial \widetilde{f}(\Omega) \cap \partial\Omega = \emptyset.$$

Since $]-\infty, a[\times\{0\} \subset \Omega$ and $\widetilde{f}(]-\infty, a[\times\{0\}) \supset]-\infty, a[\times\{0\}, \Omega \cap \widetilde{f}(\Omega) \neq \emptyset$. As $(\Theta_1 \cup v) \cap \widetilde{f}(v) = \emptyset$ and $\widetilde{f}(v) \cap \overline{\Omega}^c \neq \emptyset$, we get that $\widetilde{f}(v) \cap \overline{\Omega} = \emptyset$. And this implies that $\widetilde{f}(\Theta_1) \cap \overline{\Omega} = \emptyset$, because we are assuming that $\tilde{f}(\Theta_1) \cap \Theta_1 = \emptyset$. So, if $\widetilde{z} \in \Omega$, there is a simple continuous arc $\alpha \subset \Omega$ which connects $\widetilde{z}$ to some point $\widetilde{z}_0 \in]-\infty, a[\times\{0\}$. As $\widetilde{z}_0 \in \widetilde{f}(\Omega)$ and $\alpha \cap \left(\widetilde{f}(\Theta_1) \cup \widetilde{f}(v)\right) = \emptyset$, we get that $\alpha \subset \widetilde{f}(\Omega)$ and so $\widetilde{f}(\Omega) \supset \Omega$.

Now, suppose that $\tilde{f}(\Theta_1) = \Theta_1$. This implies that

$$\tilde{f}(\Omega) \text{ is a connected component of } \left(\Theta_1 \cup \tilde{f}(v)\right)^c,$$

and

$$]-\infty, a[\times\{0\} \subset \widetilde{f}(]-\infty, a[\times\{0\}) \subset \tilde{f}(\Omega).$$

As $(\Theta_1 \cup v) \cap \tilde{f}(v) = \emptyset$, $\tilde{f}(v)$ does not intersect $\partial\Omega$. Since both v and Ω are connected and $\tilde{f}(a, 0) \in \widetilde{f}(v)$ does not belong to Ω, we get that $\tilde{f}(v) \cap \Omega = \emptyset$.

Now, as above, let $\widetilde{z}$ be a point in Ω and α be a simple continuous arc contained in Ω connecting $\widetilde{z}$ to some $\widetilde{z}_0 \in]-\infty, a[\times\{0\}$. Since $\alpha \cap \Theta_1 = \alpha \cap \tilde{f}(v) = \emptyset$, α is contained in a connected component of $(\Theta_1 \cup \tilde{f}(v))^c$. And since $\widetilde{z}_0 \in \alpha \cap \tilde{f}(\Omega)$, it follows that $\alpha \subset \tilde{f}(\Omega)$. But this shows that any point $\widetilde{z} \in \Omega$ is a point of $\tilde{f}(\Omega)$, that is, $\Omega \subset \tilde{f}(\Omega)$. □

As Ω is open, the transitivity of $\widetilde{f}$ and the last proposition yields that it is dense in the strip. But $\Omega \subset V^-$, arriving in the final contradiction that proves theorem 2. This same argument is used often in the proofs of the next theorems.

4 Proof of theorem 1

Assume by contradiction that $\overline{p(B^-)} \neq A$. From fact 1, we know that there exists a connected component Γ of B^- that satisfies: $p(\Gamma) \subset \gamma_E^-$, $\overline{p(\Gamma)} \supset S^1 \times \{0\}$, see expression (4) and figure 6. Let $\sigma > 0$ be the number defined in the previous proof. Let $\epsilon > 0$ be sufficiently small such that:

- $S^1 \times [0, \epsilon] \subset \gamma_E^-$;
- for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times [0, \epsilon]$, $p_1 \circ \tilde{f}(\tilde{x}, \widetilde{y}) > \tilde{x} + \sigma$;

As $\overline{p(\Gamma)} \supset S^1 \times \{0\}$, there exists a sufficiently negative a such that

$$\Gamma \cap \{a\} \times [0, \epsilon] \neq \emptyset. \tag{15}$$

As in the previous proof, B^- is closed, so there must be a $\delta \leq \epsilon$ such that $(a, \delta) \in B^-$, and such that for all $0 \leq \tilde{y} < \delta, (a, \tilde{y}) \notin B^-$, that is, (a, δ) is

the "lowest" point of B^- in $\{a\} \times [0, \epsilon]$. We again denote by v the segment $\{a\} \times [0, \delta[$.

Let Γ_1 be the connected component of B^- that contains (a, δ). As $p(\Gamma_1) \cap \gamma_E^- \neq \emptyset$, we get that $\overline{p(\Gamma_1)} \subset \gamma_E^-$, something that implies the following important facts:

$$dist(p(\Gamma_1),\ S^1 \times \{1\}) > 0$$
$$dist(\Gamma_1, \mathbb{R} \times \{1\}) > 0$$

We claim that ${\Gamma_1}^+$ is not above Γ_1. We need the following propositions:

Proposition 8 *: ${\Gamma_1}^+ \cap v = \emptyset$.*

Proof:
This follows from $\Gamma^+ \subset B^-$ since, by the definition of v, $B^- \cap v = \emptyset$. □

Proposition 9 *: If $\Gamma_1 \cap \widetilde{f}(\Gamma_1) = \emptyset$ and $\Gamma_1 \prec \widetilde{f}(\Gamma_1)$, then $\widetilde{f}(v) \cap \Gamma_1 = \emptyset$.*

Proof:
As $\widetilde{f}(\Gamma_1) \subset B^-$, either $\Gamma_1 \supset \widetilde{f}(\Gamma_1)$ or $\Gamma_1 \cap \widetilde{f}(\Gamma_1) = \emptyset$. So, if $\Gamma_1 \prec \widetilde{f}(\Gamma_1)$, we get by lemma 11 that $\widetilde{f}^{-1}(\Gamma_1) \prec \Gamma_1$, so $\left[\widetilde{f}^{-1}(\Gamma_1) \cap V_a^-\right] \subset \Gamma_{1a,down}$.

On the other hand, note that $\Gamma_1 \cup v$ is a closed connected set and $(\Gamma_1 \cup v)^c$ has a connected component, denoted Ω, which contains $]-\infty, a[\times\{0\}$ and another one which contains $]a, +\infty[\times\{0\} \cup \mathbb{R} \times \{1\}$. Moreover, $\Omega \subset p^{-1}(\gamma_E^-)$. Also, it is immediate to see that

$$closure(\Omega) \supset closure(\Gamma_{1a,down}),$$

so as $\widetilde{f}^{-1}(\Gamma_1) \cap \Gamma_1 = \emptyset$ and $\left[\widetilde{f}^{-1}(\Gamma_1) \cap V_a^-\right] \subset \Gamma_{1a,down} \subset closure(\Omega)$, we have two possibilities:

i) $\widetilde{f}^{-1}(\Gamma_1) \cap v = \emptyset$, something that implies the proposition;

ii) $\widetilde{f}^{-1}(\Gamma_1) \cap v \neq \emptyset$. Consider an element $\Theta \in \left[\widetilde{f}^{-1}(\Gamma_1) \cap closure(\Omega)\right] =$ {unlimited connected components of $\widetilde{f}^{-1}(\Gamma_1) \cap closure(\Omega)$}. The connectivity of $\widetilde{f}^{-1}(\Gamma_1)$ and the fact that $\widetilde{f}^{-1}(\Gamma_1) \cap \Gamma_1 = \emptyset$ imply that Θ intersects v. As $\Theta \subset V^-$ and $\widetilde{f}(\Theta) \subset \Gamma_1$, we get that $\Theta \subset B^-$, something in contradiction with $B^- \cap v = \emptyset$. □

Moreover, if $\Gamma_1 \prec {\Gamma_1}^+$, then $\left[\widetilde{f}(\Gamma_1) \cap V_a^-\right] \subset \left[\Gamma_1^+ \cap V_a^-\right] \subset \Gamma_{1a,up}$, which implies, by the proof of lemma 8, that $\Gamma_1 \prec \widetilde{f}(\Gamma_1)$.

Lemma 14 *: $\tilde{f}(\Gamma_1)$ is not above Γ_1, that is either $\Gamma_1 \supset \widetilde{f}(\Gamma_1)$ or $\widetilde{f}(\Gamma_1) \prec \Gamma_1$, which implies that either $\Gamma_1 = {\Gamma_1}^+$, or ${\Gamma_1}^+ \prec \Gamma_1$.*

Proof:
Suppose $\Gamma_1 \prec \tilde{f}(\Gamma_1)$. As in the previous proposition, let Ω be the open connected component of $(\Gamma_1 \cup v)^c$ that is unlimited to the left and lies in $]-$

$\infty, 0[\times[0,1] \cap p^{-1}(\gamma_E^-)$. Clearly, $\partial\Omega \subset \Gamma_1 \cup v$ and as $\tilde{f}(\Gamma_1) \cap \Gamma_1 = \emptyset$, $v \cap \tilde{f}(v) = \emptyset$ and $\tilde{f}(\Gamma_1) \cap v = \tilde{f}(v) \cap \Gamma_1 = \emptyset$, we obtain

$$\tilde{f}(\partial\Omega) \cap \partial\Omega = \emptyset$$

because $\tilde{f}(\partial\Omega) = \partial\tilde{f}(\Omega) \subset \tilde{f}(\Gamma_1) \cup \tilde{f}(v)$, which is a closed connected set that does not intersect $(\Gamma_1 \cup v) \supset \partial\Omega$. As $\tilde{f}(v) \cap \overline{\Omega}^c \neq \emptyset$, because $]-\infty, a[\times\{0\} \subset \tilde{f}(]-\infty, a[\times\{0\})$, we get that $\left(\tilde{f}(\Gamma_1) \cup \tilde{f}(v)\right) \cap \overline{\Omega} = \emptyset$ and so $\Omega \subset \tilde{f}(\Omega)$. But, since Ω is open and limited to the right, this contradicts the transitivity of $\widetilde{f}$. $\Box$

So, either $\Gamma_1 = {\Gamma_1}^+$, or ${\Gamma_1}^+ \prec \Gamma_1$. Two different cases may arise.

4.1 Γ_1 is an injective component

From lemma 5, as $\overline{p(\Gamma_1)} \subset \gamma_E^-$, we obtain that $\overline{p(\Gamma_1)} \supset S^1 \times \{0\}$. So $dist(\Gamma_1, \mathbb{R} \times \{0\}) = 0$ and lemma 13 implies that Γ_1 is a down component.

Lemma 14 gives two possibilities:

1. $\tilde{f}(\Gamma_1) \subset \Gamma_1$, that is, ${\Gamma_1}^+ = \Gamma_1$
2. ${\Gamma_1}^+ \prec \Gamma_1$

If ${\Gamma_1}^+ = \Gamma_1$, then ${\Gamma_1}^+$ is also an injective down component and for all integers $k > 0$, ${\Gamma_1}^+ + (k, 0) \prec \Gamma_1$.

If ${\Gamma_1}^+ \prec \Gamma_1$, then we can prove the following:

Fact 3 *: For all integers $k > 0$, ${\Gamma_1}^+ + (k, 0) \cap \Gamma_1 = \emptyset$ and ${\Gamma_1}^+ + (k, 0) \prec \Gamma_1$.*

Proof:

If for some integer $k_0 > 0$, ${\Gamma_1}^+ + (k_0, 0)$ intersects Γ_1, then ${\Gamma_1}^+$ intersects $\Gamma_1 - (k_0, 0)$. As ${\Gamma_1}^+$ is a connected component of B^- and $\Gamma_1 - (k_0, 0) \subset B^-$ is closed and connected, we get that ${\Gamma_1}^+ \supset \Gamma_1 - (k_0, 0)$. But this contradicts ${\Gamma_1}^+ \prec \Gamma_1$ because as Γ_1 is a down component, $\Gamma_1 \prec \Gamma_1 - (k, 0)$. So for all positive integers k, ${\Gamma_1}^+ + (k, 0)$ does not intersect Γ_1. If the fact is not true, then lemma 8 implies that for some $k_* > 0$, $\Gamma_1 \prec {\Gamma_1}^+ + (k_*, 0)$, which implies that $\Gamma_1 - (k_*, 0) \prec {\Gamma_1}^+ \prec \Gamma_1$ and this again contradicts the fact that Γ_1 is a down component. $\Box$

So, in cases 1 and 2 above, for all integers $k > 0$, ${\Gamma_1}^+ + (k, 0) \cap \Gamma_1 = \emptyset$ and ${\Gamma_1}^+ + (k, 0) \prec \Gamma_1$.

The important result of this subsection is the following:

Lemma 15 *: There exists a vertical $V_r = \{r\} \times [0, 1]$ and a sequence $n_i \overset{i\to\infty}{\to} \infty$ such that $\widetilde{f}^{n_i}(\Gamma_1) \cap V_r \neq 0$ for all i.*

Proof:

As Γ_1 is a down component, $\Gamma_1 \prec \Gamma_1 - (1,0)$. Lemma 11 tell us that $\widetilde{f}(\Gamma_1) \prec \widetilde{f}(\Gamma_1) - (1,0)$. Note that $\widetilde{f}(\Gamma_1)$ may not be a whole connected component of B^-, but we will abuse notation and say that $\widetilde{f}(\Gamma_1)$ is a down component.

Above we proved that $\widetilde{f}(\Gamma_1) + (k,0) \prec \Gamma_1$ for all integers $k > 0$, so as $\widetilde{f}(\Gamma_1) \subset B^-$, in any of the possibilities 1) or 2), $\widetilde{f}(\Gamma_1) \subset \Gamma_1 \cup \Omega$ because either:

1. $\widetilde{f}(\Gamma_1) \subset \Gamma_1$
2. $\widetilde{f}(\Gamma_1) \prec \Gamma_1 \Leftrightarrow \left[\widetilde{f}(\Gamma_1) \cap V_a^-\right] \subset closure(\Gamma_{1a,down})$ (see expression (15) for a definition of a). As $closure(\Gamma_{1a,down}) \subset closure(\Omega)$, which is a connected set (see the proof of proposition 9 for a definition of Ω) and $\partial\Omega \subset \Gamma_1 \cup v$ does not intersect $\widetilde{f}(\Gamma_1)$, we get that $\widetilde{f}(\Gamma_1) \subset \Omega$.

Let us fix some $k' > 0$ in a way that $\widetilde{f}(\Gamma_1) + (k',0)$ intersects $v = \{a\} \times [0,\delta[$, see (15). The reason why such a k' exists is the following: As $\widetilde{f}(\Gamma_1) + (k,0) \cap \Gamma_1 = \emptyset$ and $\widetilde{f}(\Gamma_1) + (k,0) \prec \Gamma_1$ for all integers $k > 0$, we get that $\left[\widetilde{f}(\Gamma_1) + (k,0) \cap V_a^-\right] \subset closure(\Gamma_{1a,down}) \subset closure(\Omega)$. And as $\partial\Omega \subset \Gamma_1 \cup v$ and $\overline{\Omega} \subset]-\infty,0] \times [0,1]$, we get that if $k' > 0$ is sufficiently large in a way that $\widetilde{f}(\Gamma_1)+(k',0)$ intersects $\{1\}\times[0,1]$, then $\widetilde{f}(\Gamma_1)+(k',0)$ intersects the boundary of Ω in the only possible place, v. Denote by Γ^* an unlimited connected component of $\widetilde{f}(\Gamma_1) + (k',0) \cap closure(\Omega)$. By the choice of $k' > 0$ and the connectivity of $\widetilde{f}(\Gamma_1)+(k',0)$, we get that Γ^* is not contained in B^- because it intersects v. So, there exists a positive integer $a_1 > 0$ such that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0,+\infty[\times[0,1]$. Remember that $\widetilde{f}(\Gamma_1)+(k',0) \prec \Gamma_1$ and, as we said above, either $\widetilde{f}(\Gamma_1) \subset \Gamma_1$ or $\widetilde{f}(\Gamma_1) \prec \Gamma_1$. In case $\widetilde{f}(\Gamma_1) \subset \Gamma_1$, we get that $\widetilde{f}^{a_1+1}(\Gamma_1) \subset \Gamma_1$. Before continuing the proof, let us state the following:

Proposition 10 *: If Γ is a connected component of B^- which satisfies $\widetilde{f}(\Gamma) \cap \Gamma = \emptyset$ and $\widetilde{f}(\Gamma) \prec \Gamma$, then $\widetilde{f}^n(\Gamma) \cap \Gamma = \emptyset$ and $\widetilde{f}^n(\Gamma) \prec \Gamma$ for all integers $n > 0$.*

Proof:

By contradiction, suppose there exists some $n_0 > 1$ (then smallest one) such that $\widetilde{f}^{n_0}(\Gamma) \cap \Gamma \neq \emptyset$. This means that Γ, $\widetilde{f}(\Gamma), \widetilde{f}^2(\Gamma), ..., \widetilde{f}^{n_0-1}(\Gamma)$ are disjoint closed connected subsets of the strip $\widetilde{A}$, each of them having a connected complement and $\widetilde{f}^{n_0}(\Gamma) \subset \Gamma$. As $\widetilde{f}(\Gamma_1) \prec \Gamma_1$, lemmas 10 and 11 imply that

$$\widetilde{f}^{n_0-1}(\Gamma) \prec ... \prec \widetilde{f}^2(\Gamma) \prec \widetilde{f}(\Gamma) \prec \Gamma. \tag{16}$$

On the other hand, as $\widetilde{f}^{n_0}(\Gamma) \cap \widetilde{f}^{n_0-1}(\Gamma) = \emptyset$, lemma 11 implies that $\widetilde{f}^{n_0}(\Gamma) \prec \widetilde{f}^{n_0-1}(\Gamma)$. So, $\widetilde{f}^{n_0}(\Gamma) \cap \left(\widetilde{f}^{n_0-1}(\Gamma)\right)_{down}$ has an unlimited connected component. As $\widetilde{f}^{n_0}(\Gamma) \subset \Gamma$ and $\widetilde{f}^{n_0-1}(\Gamma) \cap \Gamma = \emptyset$, using lemma 7 we get that $\Gamma \prec \widetilde{f}^{n_0-1}(\Gamma)$, a contradiction with expression (16). So, $\widetilde{f}^n(\Gamma) \cap \Gamma = \emptyset$ for all integers $n > 0$. The other implication follows from lemma 11. $\square$

So, if $\widetilde{f}(\Gamma_1) \prec \Gamma_1$ ($\widetilde{f}(\Gamma_1) \cap \Gamma_1 = \emptyset$), then $\widetilde{f}^{a_1+1}(\Gamma_1) \cap \Gamma_1 = \emptyset$ and $\widetilde{f}^{a_1+1}(\Gamma_1) \prec \Gamma_1$. As Γ_1 is a down component, $\widetilde{f}^{a_1}(\widetilde{f}(\Gamma_1) + (k', 0)) = \widetilde{f}^{a_1+1}(\Gamma_1) + (k', 0) \prec \widetilde{f}^{a_1+1}(\Gamma_1)$. Now, note that $\widetilde{f}^{a_1+1}(\Gamma_1) + (k', 0) \cap \Gamma_1 = \emptyset$.

This happens because, if $\widetilde{f}^{a_1+1}(\Gamma_1) + (k', 0) \cap \Gamma_1 \neq \emptyset$, then there would be a connected component of B^-, denoted Ψ, containing both $\widetilde{f}^{a_1+1}(\Gamma_1)$ and $\Gamma_1 - (k', 0)$. Clearly, Ψ is not Γ_1 and both $\Psi \cap \Gamma_{1down}$ and $\Psi \cap \Gamma_{1up}$ have unlimited connected components, because $\widetilde{f}^{a_1+1}(\Gamma_1) \prec \Gamma_1$ and $\Gamma_1 \prec \Gamma_1 - (k', 0)$, a contradiction. Thus, $\widetilde{f}^{a_1}(\widetilde{f}(\Gamma_1) + (k', 0)) = \widetilde{f}^{a_1+1}(\Gamma_1) + (k', 0) \prec \Gamma_1$ and so, as

$$\Gamma^* \subset \widetilde{f}(\Gamma_1) + (k', 0) \ \Rightarrow \ \widetilde{f}^{a_1}(\Gamma^*) \cap \Gamma_1 = \emptyset \text{ and } \widetilde{f}^{a_1}(\Gamma^*) \prec \Gamma_1.$$

As above, the fact that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0, +\infty[\times[0, 1]$ implies that $\widetilde{f}^{a_1}(\Gamma^*) \cap closure(\Omega)$ has an unlimited connected component, Γ^{**} which intersects v. So, Γ^{**} is not contained in B^- and thus there exists an integer $a_2 > 0$ such that $\widetilde{f}^{a_2}(\Gamma^{**}) \subset \widetilde{f}^{a_2+a_1+1}(\Gamma_1) + (k', 0)$ intersects $]0, +\infty[\times[0, 1]$. In exactly the same way as above, we obtain an unlimited connected component of $\widetilde{f}^{a_2}(\Gamma^{**}) \cap closure(\Omega)$, denoted Γ^{***} which intersects v. So, Γ^{***} is not contained in B^- and there exists an integer $a_3 > 0$ such that $\widetilde{f}^{a_3}(\Gamma^{***})$ intersects $]0, +\infty[\times[0, 1]$ and so on.

Thus, if we define $n_i = a_1 + a_2 + ... + a_i + 1$, we get that $n_i \overset{i\to\infty}{\to} \infty$ and for all $i \geq 1$, $\widetilde{f}^{n_i-1}(\widetilde{f}(\Gamma_1) + (k', 0)) \supset \widetilde{f}^{n_i-1}(\Gamma^*) = \widetilde{f}^{a_2+...+a_i}(\widetilde{f}^{a_1}(\Gamma^*)) \supset \widetilde{f}^{a_2+...+a_i}(\Gamma^{**}) \supset ... \supset \widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*...*}})$ and $\widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*...*}})$ intersects $]0, +\infty[\times[0, 1]$. So,

$$\widetilde{f}^{n_i}(\Gamma_1) \text{ intersects } V_0 - (k', 0) = V_{-k'}$$

and the lemma is proved. □

But the lemma implies that $\omega(B^-) \neq \emptyset$, contradicting theorem 2. Therefore we must have that Γ_1 is a non-injective component.

4.2 Γ_1 is an non-injective component

From lemma 5, as $\overline{p(\Gamma_1)} \subset \gamma_E^-$, we obtain that $\overline{p(\Gamma_1)} \supset S^1 \times \{0\}$.

The next result is interesting by itself:

Fact 4 *: If Γ is a non-injective component of B^- and $\widetilde{f}(\Gamma) \subset \Gamma$, then there exists an integer $k > 0$ such that $\widetilde{f}^{-1}(\Gamma) \subset \Gamma + (k, 0)$.*

Proof:

Since $\tilde{f}(\Gamma) \subset \Gamma$, we get $\Gamma \subset \tilde{f}^{-1}(\Gamma)$. As $\Gamma \subset V^-$, $\tilde{f}^{-1}(\Gamma)$ is limited to the right, so there exists an integer $k > 0$ such that $\tilde{f}^{-1}(\Gamma) - (k, 0) \subset V^-$. If $i \geq 1$,

$$\tilde{f}^i\left(\tilde{f}^{-1}(\Gamma) - (k, 0)\right) = \tilde{f}^{i-1}(\Gamma) - (k, 0) \subset \Gamma - (k, 0) \subset \Gamma.$$

So, as the closed connected set $\tilde{f}^{-1}(\Gamma) - (k, 0)$ is unlimited to the left and has all its positive iterates in V^-, it is contained in B^-. As $\Gamma \subset \tilde{f}^{-1}(\Gamma) \Rightarrow$

$\Gamma-(k,0)\subset \tilde{f}^{-1}(\Gamma)-(k,0)$. So, $\tilde{f}^{-1}(\Gamma)-(k,0)$ intersects Γ because $\Gamma\supset\Gamma-(k,0)$. But Γ is a connected component of B^- and $\tilde{f}^{-1}(\Gamma)-(k,0)$ is connected, therefore

$$\tilde{f}^{-1}(\Gamma)-(k,0)\subset\Gamma,$$

something that proves the fact. $\Box$

Clearly, for any integer $n\geq 1$,

$$\widetilde{f}^{-n}(\Gamma)\subset\Gamma+(n.k,0).$$

In contrast with the case when Γ_1 is injective, lemma 14 implies that the only possibility here is $\widetilde{f}(\Gamma_1)\prec\Gamma_1$ because of the next lemma:

Lemma 16 *: It is not possible that* $\widetilde{f}(\Gamma_1)\subset\Gamma_1$.

Proof:
Suppose that $\widetilde{f}(\Gamma_1)\subset\Gamma_1$. Let

$$\Omega_1=\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(\Omega),$$

where, as in the proofs of proposition 9 and lemma 14, Ω is the open connected component of $(\Gamma_1\cup v)^c$ that contains $]-\infty,a[\times\{0\}$. Note that Ω is contained in $p^{-1}(\gamma_E^-)$ and clearly, $\widetilde{f}^{-1}(\Omega_1)\subset\Omega_1$. Moreover, the following is true:

Proposition 11 *:* Ω_1 *is contained in* $p^{-1}(\gamma_E^-)$.

Before proving this proposition, let us show how it is used to prove our lemma. Since Ω_1 is open and $\widetilde{f}^{-1}(\Omega_1)\subset\Omega_1$, we must have, by the transitivity of $\widetilde{f}$, that Ω_1 is dense. But this contradicts the proposition. $\Box$

Proof of proposition 11:
First note that, as the boundary of Ω is contained in $\Gamma_1\cup v$, for all integers $i>0$ we have:

$$\partial\left(\widetilde{f}^{-i}(\Omega)\right)\subset\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(\Gamma_1\cup v)=\left(\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(\Gamma_1)\right)\cup\left(\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(v)\right). \tag{17}$$

Clearly Ω_1 is an open set. Let us show that it is connected. Each set of the form $\widetilde{f}^{-i}(\Omega)$ is connected because $\widetilde{f}$ is a homeomorphism. Also, since $\widetilde{f}^{-i}(]-\infty,a[\times\{0\})\subset]-\infty,a[\times\{0\}$, we have $\widetilde{f}^{-i}(\Omega)\cap\Omega\neq\emptyset$. But Ω is also open and connected, so Ω_1 must be connected.

For all integers $i>0$, as $\widetilde{f}^{-i}(\Omega)$ is connected, intersects $\mathbb{R}\times\{0\}$ and is disjoint from $\mathbb{R}\times\{1\}$, if we show that $\left(\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(\Gamma_1)\right)\cup\left(\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(v)\right)\subset p^{-1}(\gamma_E^-)$, then expression (17) implies that $\widetilde{f}^{-i}(\Omega)\subset p^{-1}(\gamma_E^-)$, which gives: $\Omega_1\subset p^{-1}(\gamma_E^-)$ and the proof is complete.

Let us analyze first what happens to $\widetilde{f}^{-n}(\Gamma_1)$, for all integers $n>0$.

From fact 4, $\bigcup_{n=0}^{\infty}\widetilde{f}^{-n}(\Gamma_1)\subset\bigcup_{n=0}^{\infty}(\Gamma_1+(n,0))\subset p^{-1}(\gamma_E^-)$.

We are left to deal with $\bigcup_{n=0}^{\infty} \widetilde{f}^{-n}(v)$. Let us show that $\widetilde{f}^{-1}(v) \subset \Omega$. From the choice of v, $\widetilde{f}^{-1}(v) \cap v = \emptyset$. Also, from the definition of $v = \{a\} \times [0, \delta[$, we get that $\widetilde{f}^{-1}(v) \cap \Gamma_1 = \emptyset$ because $v \cap B^- = \emptyset$. Finally, the following inclusions

$$\Omega \supset]-\infty, a[\times\{0\} \text{ and }]-\infty, a[\times\{0\} \supset \widetilde{f}^{-1}(]-\infty, a[\times\{0\})$$

imply that $\widetilde{f}^{-1}(v) \cap \Omega \neq \emptyset \Rightarrow \widetilde{f}^{-1}(v) \subset \Omega \subset p^{-1}(\gamma_E^-)$.

So, $\widetilde{f}^{-2}(v) \subset \widetilde{f}^{-1}(\Omega)$, whose boundary, $\partial\left(\widetilde{f}^{-1}(\Omega)\right)$, is contained in $\left(\bigcup_{n=0}^{\infty} (\Gamma_1 + (n, 0))\right) \cup \widetilde{f}^{-1}(v) \subset p^{-1}(\gamma_E^-)$. As above, as $\widetilde{f}^{-1}(\Omega)$ is connected, intersects $\mathbb{R} \times \{0\}$ and is disjoint from $\mathbb{R} \times \{1\}$, we get that

$$\widetilde{f}^{-2}(v) \subset \widetilde{f}^{-1}(\Omega) \subset p^{-1}(\gamma_E^-).$$

So, $\widetilde{f}^{-3}(v) \subset \widetilde{f}^{-2}(\Omega)$ and an analogous argument implies that $\widetilde{f}^{-3}(v) \subset \widetilde{f}^{-2}(\Omega) \subset p^{-1}(\gamma_E^-)$. An induction shows that

$$\widetilde{f}^{-n}(v) \subset \widetilde{f}^{-n+1}(\Omega) \subset p^{-1}(\gamma_E^-) \text{ for all integers } n \geq 1,$$

and the proposition is proved. $\square$

Thus, if Γ_1 is a non-injective component, the only possibility is $\tilde{f}(\Gamma_1) \prec \Gamma_1$. The next lemma is a version of lemma 15 to a non-injective Γ_1 :

Lemma 17 *: There exists a vertical $V_r = \{r\} \times [0, 1]$ and a sequence $n_i \overset{i \to \infty}{\to} \infty$ such that $\widetilde{f}^{n_i}(\Gamma_1) \cap V_r \neq 0$ for all integer $i \geq 1$.*

Proof:

As $\widetilde{f}(\Gamma_1) \prec \Gamma_1 \Rightarrow \left[\widetilde{f}(\Gamma_1) \cap V_a^-\right] \subset closure(\Gamma_{1a,down})$ (see expression (15) for a definition of a). As $closure(\Gamma_{1a,down}) \subset closure(\Omega)$, which is a connected set and $\partial\Omega \subset \Gamma_1 \cup v$ does not intersect $\widetilde{f}(\Gamma_1)$, we get that $\widetilde{f}(\Gamma_1) \subset \Omega$.

If for some integers $n_0 > 0$ and $k_0 > 0$, $\widetilde{f}^{n_0}(\Gamma_1) + (k_0, 0) \cap \Gamma_1 \neq \emptyset$, then $\widetilde{f}^{n_0}(\Gamma_1) \cap \Gamma_1 - (k_0, 0) \neq \emptyset \Rightarrow \widetilde{f}^{n_0}(\Gamma_1) \cap \Gamma_1 \neq \emptyset \Rightarrow \widetilde{f}^{n_0}(\Gamma_1) \subset \Gamma_1$, which, using proposition 10, implies that $\widetilde{f}(\Gamma_1) \subset \Gamma_1$, a contradiction. So, for all integers $n > 0$ and $k > 0$, as $\widetilde{f}^n(\Gamma_1) + (k, 0) \supset \widetilde{f}^n(\Gamma_1)$ and

$$\tilde{f}^n(\Gamma_1) \prec ... \prec \tilde{f}(\Gamma_1) \prec \Gamma_1 \text{ (see proposition 10)},$$

we get that $\widetilde{f}^n(\Gamma_1) + (k, 0) \cap \Gamma_1 = \emptyset$ and $\widetilde{f}^n(\Gamma_1) + (k, 0) \prec \Gamma_1$.

Now the proof goes exactly as in lemma 15. $\square$

Of course, we have arrived at the same contradiction as in subsection 4.1, and so theorem 1 is proved.

5 Proof of theorem 3

Lemma 18 *: There exists an integer $N_1 > 0$ such that $\tilde{f}^{N_1}(B^-) \subset B^- - (1,0)$*

Proof:
Theorem 2 shows that $\omega(B^-) = \emptyset$, so there must be an integer $N_1 > 0$ such that, for all $n \geq N_1$, $\tilde{f}^n(B^-) \subset]-\infty, -1[\times[0,1]$. Suppose, by contradiction, that there exists $\widetilde{z} \in B^-$ such that $\tilde{f}^{N_1}(\widetilde{z}) + (1,0) \notin B^-$.

Let Γ be the connected component of B^- that contains $\widetilde{z}$. Clearly $\tilde{f}^{N_1}(\Gamma) \subset V_{-1}^-$. As $f^{N_1}(\widetilde{z}) + (1,0) \in \tilde{f}^{N_1}(\Gamma) + (1,0)$, $\tilde{f}^{N_1}(\Gamma) + (1,0)$ is not a subset of B^-. But $\tilde{f}^{N_1}(\Gamma) + (1,0)$ is connected, unbounded and $\tilde{f}^{N_1}(\Gamma) + (1,0) \subset V^-$, therefore there must be a $N_2 > 0$ such that $\tilde{f}^{N_2}(\tilde{f}^{N_1}(\Gamma) + (1,0)) = \tilde{f}^{N_1+N_2}(\Gamma) + (1,0)$ is not contained in V^-. This implies that $f^{N_1+N_2}(\Gamma)$ is not contained in V_{-1}^-, a contradiction that proves the lemma. $\Box$

As $\tilde{f}^{N_1}(B^-) \subset B^- - (1,0)$, for any positive integer k,

$$\tilde{f}^{kN_1}(B^-) \subset B^- - (k,0) \subset V_{-k}^-,$$

and so it follows that, for any point $\widetilde{z} \in B^-$,

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})}{n} \leq -\frac{1}{N_1},$$

and this proves theorem 3.

6 Proof of theorem 4

Let $\epsilon > 0$ be such that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times \{[0,\epsilon] \cup [1-\epsilon, 1]\}$, $p_1 \circ \tilde{f}(\tilde{x}, \widetilde{y}) > \tilde{x} + \sigma$, for a certain fixed $\sigma > 0$.

As theorem 1 says that $\overline{p(B^-)} = A$, there exists a sufficiently negative b such that

$$\Theta \cap \{b\} \times [0,\epsilon] \neq \emptyset,$$

for some connected component Θ of B^-. As in the beginning of the proof of theorem 1, in the following we will consider the "lowest" component of B^- in $\{b\} \times [0,\epsilon]$.

First, remember that as B^- is closed, there must be a $0 < \delta \leq \epsilon$ such that $(b,\delta) \in B^-$, and for all $0 \leq \tilde{y} < \delta, (b,\tilde{y}) \notin B^-$, that is, (b,δ) is the "lowest" point of B^- in $\{b\} \times [0,\epsilon]$. We denote by v the segment $\{b\} \times [0,\delta[$.

Let Γ_1 be the connected component of B^- that contains (b,δ). By propositions 8, 9 and lemma 14, if $\Gamma_1 \prec \tilde{f}(\Gamma_1)$, then the set Ω, which is the open connected component of $(\Gamma_1 \cup v)^c$ that is unlimited to the left, lies in $]-\infty, 0[\times[0,1]$ and contains $]-\infty, b[\times\{0\}$ satisfies the following: $\Omega \subset \tilde{f}(\Omega) \Leftrightarrow \widetilde{f}^{-1}(\Omega) \subset \Omega$ and this contradicts the existence of a dense orbit for $\widetilde{f}$.

So, either $\widetilde{f}(\Gamma_1) \subset \Gamma_1$ or $\widetilde{f}(\Gamma_1) \prec \Gamma_1$. In order to analyze the two previous possibilities, we have to consider all possible "shapes" for Γ_1 :

1) Γ_1 is an injective down component;

2) Γ_1 is an injective up component;

3) Γ_1 is a non-injective component;

In case 1, if $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or $\tilde{f}(\Gamma_1) \prec \Gamma_1$ and in case 3, if $\tilde{f}(\Gamma_1) \prec \Gamma_1$, lemmas 15 and 17 imply that $\omega(B^-)$ (see (2)) is not empty. And this is a contradiction with theorem 2.

So, either Γ_1 is an injective up component or Γ_1 is an non-injective component and $\tilde{f}(\Gamma_1) \subset \Gamma_1$. Suppose that Γ_1 is an injective up component. We have two possibilities:

I) $dist(\Gamma_1, \mathbb{R} \times \{1\}) > 0$;

II) $dist(\Gamma_1, \mathbb{R} \times \{1\}) = 0$;

Lemma 19 *: If Γ_1 is an injective up component, then $dist(\Gamma_1, \mathbb{R} \times \{1\}) = 0$.*

Proof:

As Γ_1 is an injective up component, lemma 13 implies that

$$dist(\Gamma_1, \mathbb{R} \times \{0\}) > 0.$$

So if I) holds, there exists $\epsilon_1 > 0$ such that $\Gamma_1 \cap \mathbb{R} \times \{[0, \epsilon_1] \cup [1 - \epsilon_1, 1]\} = \emptyset$.

Since $\tilde{f}$ is transitive, f is transitive and thus there is a point $z \in S^1 \times [1 - \epsilon_1/2, 1]$ and an integer $n > 0$ such that $f^{-n}(z) \in S^1 \times [0, \epsilon_1/2]$. We know that $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or $\tilde{f}(\Gamma_1) \prec \Gamma_1$, so by proposition 10 and lemmas 10 and 11 we get that

$$\tilde{f}^n(\Gamma_1) \subset \Gamma_1 \text{ } or \text{ } \tilde{f}^n(\Gamma_1) \prec \Gamma_1.$$

Now let $d \in \mathbb{R}$ be such that $\widetilde{f}^{-i}(V_d) \subset V^-_{m_{\Gamma_1}-1}$ (see expression (5)) for $i = 0, 1, ..., n$ and $\tilde{f}^n(\Gamma_1) \cap V_d^- \subset \Gamma_1 \cup \Gamma_{1\ down}$, see proposition 5.

Let $\widetilde{z} \in V_d^-$ be a point such that $p(\widetilde{z}) = z$ and let k be the vertical line segment that has as extremes $\widetilde{z}$ and a point $\widetilde{z}_1$ in $\mathbb{R} \times \{1\}$.

As $\widetilde{f}^{-n}(\widetilde{z}) \in \mathbb{R} \times [0, \epsilon_1/2] \cap V^-_{m_{\Gamma_1}-1}$, we obtain that $\widetilde{f}^{-n}(\widetilde{z}) \in \Gamma_{1\ down}$. As $\widetilde{f}^{-n}(V_d) \subset V^-_{m_{\Gamma_1}-1}$, we get that $\widetilde{f}^{-n}(k) \cap V_{m_{\Gamma_1}} = \emptyset$. Since $\tilde{f}^{-n}(\widetilde{z}_1) \notin \Gamma_{1\ down}$ and $\tilde{f}^{-n}(\tilde{z}) \in \Gamma_{1\ down}$, and since k is connected, $\tilde{f}^{-n}(k) \cap \partial(\Gamma_{1\ down}) \neq \emptyset$. But $\partial(\Gamma_{1\ down}) \subset \Gamma_1 \cup V_{m_{\Gamma_1}}$ and as $\widetilde{f}^{-n}(k) \cap V_{m_{\Gamma_1}} = \emptyset$, we get that $\widetilde{f}^{-n}(k) \cap \Gamma_1 \neq \emptyset$, which implies that $k \cap \widetilde{f}^n(\Gamma_1) \neq \emptyset$ and this is a contradiction because $k \subset V_d^- \cap \Gamma_{1\ up}$ and $\tilde{f}^n(\Gamma_1) \cap V_d^- \subset \Gamma_1 \cup \Gamma_{1\ down}$. So I) does not hold. $\square$

Thus if Γ_1 is injective, then II) holds. So consider a sufficiently negative c such that

$$\Gamma_1 \cap \{c\} \times [1 - \epsilon, 1] \neq \emptyset,$$

where $\epsilon > 0$ was defined in the beginning of this section. As above, as B^- is closed, there must be a $0 < \mu \leq \epsilon$ such that $(c, 1 - \mu) \in B^-$, and for all $1 - \mu \leq \tilde{y} < 1, (c, \tilde{y}) \notin B^-$, that is, $(c, 1 - \mu)$ is the "highest" point of B^- in $\{c\} \times [1 - \epsilon, 1]$. We denote by w the segment $\{c\} \times]1 - \mu, 1]$.

Let Γ_2 be the connected component of B^- that contains $(c, 1-\mu)$. An argument analogous to the one which implies that Γ_1 can not be an injective down component, implies that Γ_2 can not be an injective up component, so if Γ_1 is injective, $\Gamma_1 \neq \Gamma_2$ and thus $\Gamma_1 \cap \Gamma_2 = \emptyset$. So Γ_2 is either non-injective or an injective down component. In the second case, as $dist(\Gamma_2, \mathbb{R} \times \{1\}) > 0$ (see lemma 13), it is not possible that $\Gamma_1 \prec \Gamma_2$. But this implies that $\Gamma_2 \prec \Gamma_1$ and so Γ_2 intersects v. And this is a contradiction with the definition of v. So Γ_2 is a non-injective component. By exactly the same reasoning applied to Γ_1, we must have $\tilde{f}(\Gamma_2) \subset \Gamma_2$.

The following lemma concludes the proof of theorem 4, because either Γ_1 is an non-injective component and $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or, in case Γ_1 is an injective up component, Γ_2 is non-injective and $\tilde{f}(\Gamma_2) \subset \Gamma_2$.

Lemma 20 *: If Γ is a non-injective component of B^- such that $\tilde{f}(\Gamma) \subset \Gamma$, then $\overline{p(\Gamma)} = A$.*

Proof:

First of all, note that the set Γ has all the properties required for the set D in subsection 2.2, so lemma 3 implies that either $\overline{p(\Gamma)} \supset S^1 \times \{0\}$ or $\overline{p(\Gamma)} \supset S^1 \times \{1\}$. So let us suppose, without loss of generality, that

$$\overline{p(\Gamma)} \supset S^1 \times \{0\}.$$

Lemma 4 shows that, if $p(\Gamma)$ is not dense in A, then there exists a simple closed curve $\gamma \subset interior(A)$, which is homotopically non trivial and such that $\overline{p(\Gamma)} \cap \gamma = \emptyset$. But since $p(\Gamma)$ is connected, we must have $\Gamma \subset p^{-1}(\gamma^-)$.

As Γ is closed and $S^1 \times \{0\} \subset \overline{p(\Gamma)}$, we can find a point $(c', \delta') \in \Gamma$ such that:

1. $\delta' < \epsilon$, where $\epsilon > 0$ was defined in the beginning of this section;
2. if $v' = c' \times [0, \delta'[$, then $\Gamma \cap v' = \emptyset$ and $\overline{v'} \subset p^{-1}(\gamma^-)$.

Now, let us choose Ω' as the connected component of $(\Gamma \cup v')^c$ that contains $]-\infty, c'[\times\{0\}$ and consider the following set, as we did in proposition 11:

$$\Omega_{sat} = \bigcup_{n=0}^{\infty} \tilde{f}^{-n}(\Omega')$$

A simple repetition of the same arguments used in the proof of proposition 11 yields that $\Omega_{sat} \subset p^{-1}(\gamma^-)$. Again, since $\tilde{f}^{-1}(\Omega_{sat}) \subset \Omega_{sat}$ this contradicts the transitivity of $\tilde{f}$ and finishes the proof. $\Box$

As we know that at least one member of the set $\{\Gamma_1, \Gamma_2\}$ is non-injective and positively invariant, the above lemma implies that Γ_1 or Γ_2 must have a dense projection to the annulus. One more thing can be said, which will be important in the proof of the next theorem:

Proposition 12 *: Both Γ_1 and Γ_2 are non-injective components.*

Proof:

If the proposition is not true for Γ_1, then as we already proved, Γ_1 must be an injective up component. As Γ_2 does not cross v and Γ_1 does not cross w, by the definitions of v and w, it must be the case that $\Gamma_1 \prec \Gamma_2$ and so

$$dist(\Gamma_2, \mathbb{R} \times \{0\}) > 0,$$

something that contradicts lemma 20. So Γ_1 is a non-injective component. To conclude the proof we have to note that Γ_1 and Γ_2 have analogous properties, Γ_1 is the connected component of B^- that contains the lowest point of B^- in $\{b\} \times [0, \epsilon]$ and Γ_2 is the connected component of B^- that contains the "highest" point of B^- in $\{c\} \times [1-\epsilon, 1]$. So Γ_2 must also be a non-injective component. $\Box$

Summarizing, the above results prove that, for every vertical segment u of the form $\{l\} \times [0, \epsilon]$ (or $\{l\} \times [1-\epsilon, 1]$) which intersects B^- (see the definition of $\epsilon > 0$ in the beginning of this section), the "lowest" (or "highest") component of B^- in u must be non-injective, $\widetilde{f}$-positively invariant and dense when projected to A.

7 Proof of theorem 5

Without loss of generality, suppose $\Gamma \subset B^-$ is an injective down connected component. Consider a vertical $v = \{c\} \times [0, \epsilon[$, such that:

$$\begin{array}{l} 1)\ 0 < \epsilon \leq \epsilon',\ \text{where } \epsilon' \text{ is such that } \forall (\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times [0, \epsilon'], \\ p_1 \circ \tilde{f}(\tilde{x}, \widetilde{y}) > \tilde{x} + \sigma,\ \text{for some } \sigma > 0; \\ 2)\ v \subset \Gamma_{down}; \\ 3)\ v \cap B^- \neq \emptyset. \end{array} \tag{18}$$

The above is possible because $\overline{p(B^-)} = A$ and if $\{c\} \times [0, b[\subset \Gamma_{down}$ and $(c, b) \in \Gamma$, then $B^- \cap \{c\} \times [0, b[\neq \emptyset$, see the proof of the previous theorem. So we can choose a sufficiently small c such that $\{c\} \times [0, \epsilon'] \cap B^- \neq \emptyset$, where ϵ' comes from 1) of (18). If $\{c\} \times [0, \epsilon'[\subset \Gamma_{down}$, we are done. If not, let $0 < \epsilon < \epsilon'$ be such that $v = \{c\} \times [0, \epsilon[\subset \Gamma_{down}$ and $(c, \epsilon) \in \Gamma$.

Denote by Θ the lowest connected component of B^- in v and by $w = \{c\} \times [0, \delta[\subset v$ the vertical such that $w \cap B^- = \emptyset$ and $(c, \delta) \in \Theta$. From theorem 4 we know that Θ satisfies the following conditions:

$$\begin{array}{l} i)\ \Theta \text{ is non-injective}; \\ ii)\ \widetilde{f}(\Theta) \subset \Theta; \\ iii)\ \overline{p(\Theta)} = A; \end{array} \tag{19}$$

If $\Theta \prec \Gamma$, then proposition 5 implies the existence of a real number d such that $\Theta \cap V_d^- \subset \Gamma_{down}$. As $dist(\Gamma, \mathbb{R} \times \{1\}) > 0$, we get that $dist(\Theta, \mathbb{R} \times \{1\}) > 0$, something that contradicts property iii) of expression (19).

So we can assume that $\Gamma \prec \Theta$. As in most of the previous results, let Ω be the connected component of $(\Theta \cup w)^c$ that contains $]-\infty, c[\times\{0\}$. We know that

$$closure(\Omega) \supset closure(\Theta_{c,down}) \tag{20}$$

and, as $\Gamma \prec \Theta$, $[\Gamma \cap V_c^-] \subset \Theta_{c,down}$. So, using expression (20) we get that $\Gamma \subset \Omega$ because Γ is connected, $\Gamma \cap \Omega \neq \emptyset$ and $\Gamma \cap \partial\Omega \subset \Gamma \cap (\Theta \cup w) = \emptyset$. The rest of our proof will be divided in two steps:

Step 1: Here we are going to prove that for all integers $n > 0$ and $k \geq 0$, $\widetilde{f}^n(\Gamma) + (k, 0)$ is disjoint from Θ and $\widetilde{f}^n(\Gamma) \subset \Omega \Rightarrow \widetilde{f}^n(\Gamma) + (k, 0) \prec \Theta$.

As $\Gamma \prec \Theta$, we get for any integer $n > 0$, that either $\widetilde{f}^n(\Gamma) \subset \Theta$ or $\widetilde{f}^n(\Gamma) \cap \Theta = \emptyset$ and $\widetilde{f}^n(\Gamma) \prec \Theta$, which implies that $\widetilde{f}^n(\Gamma) \subset \Omega$ because $\widetilde{f}^n(\Gamma) \subset B^-$. To begin, suppose $\widetilde{f}(\Gamma) \subset \Theta$. This means that $\widetilde{f}^{-1}(\Theta) \supset \Gamma$ and so, if $\widetilde{f}^{-1}(\Theta) \cap V = \emptyset$, then $\widetilde{f}^{-1}(\Theta)$ is contained in a connected component of B^-, that is, $\widetilde{f}^{-1}(\Theta) = \Gamma$, a contradiction because Γ is injective and Θ is not. So, $\widetilde{f}^{-1}(\Theta) \cap V \neq \emptyset$. Let Γ' be the connected component of $\widetilde{f}^{-1}(\Theta) \cap V^-$ that contains Γ. The fact that $\widetilde{f}^{-1}(\Theta)$ is connected implies that Γ' intersects V, is contained in B^- and contains Γ. So, $\Gamma' = \Gamma$ and this is a contradiction because $\Gamma \subset \Omega$ and $\Omega \cap V = \emptyset$. So, $\widetilde{f}(\Gamma) \cap \Theta = \emptyset \Rightarrow \widetilde{f}(\Gamma) \prec \Theta \Rightarrow \widetilde{f}(\Gamma) \subset \Omega$.

Now note that, $\widetilde{f}(\Gamma) = \Gamma^+$, because, if this is not the case, then $\widetilde{f}^{-1}(\Gamma^+) \supset \Gamma$ is not a connected component of B^-, so $\widetilde{f}^{-1}(\Gamma^+) \cap V \neq \emptyset$, which means that $\widetilde{f}^{-1}(\Gamma^+) \cap w \neq \emptyset$ because $\widetilde{f}^{-1}(\Gamma^+) \cap \Theta = \emptyset$ and $\widetilde{f}^{-1}(\Gamma^+) \cap \Omega \neq \emptyset$. If we denote by Γ^{+*} the connected component of $\widetilde{f}^{-1}(\Gamma^+) \cap \Omega$ that contains Γ, then as $\widetilde{f}^{-1}(\Gamma^+)$ is connected, Γ^{+*} intersects w and is contained in B^-, a contradiction.

So, $\widetilde{f}(\Gamma) = \Gamma^+ \subset \Omega$ and an induction using the above argument implies that for every integer $n > 0$:

1) $\widetilde{f}^n(\Gamma) \cap \Theta = \emptyset$;
2) $\widetilde{f}^n(\Gamma)$ is a connected component of B^-;
3) $\widetilde{f}^n(\Gamma) \subset \Omega$;
4) $\widetilde{f}^n(\Gamma) \prec \Theta$;

As $\Gamma \subset B^-$ is an injective down connected component, the same holds for $\widetilde{f}^n(\Gamma)$ (for any integer $n > 0$). So the assertion from step 1 holds.

Step 2: Here we perform the same construction as we did in lemma 15, see it for more details.

Let us fix some $k' > 0$ in a way that $\widetilde{f}(\Gamma) + (k', 0)$ intersects w. Denote by Γ^* an unlimited connected component of $\widetilde{f}(\Gamma) + (k', 0) \cap closure(\Omega)$. By the choice of $k' > 0$ and the connectivity of $\widetilde{f}(\Gamma) + (k', 0)$, we get that Γ^* is not contained in B^- because it intersects v. So, there exists a positive integer $a_1 > 0$ such that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0, +\infty[\times[0, 1]$.

Step 1 implies that $\widetilde{f}^{a_1}(\widetilde{f}(\Gamma) + (k', 0)) = \widetilde{f}^{a_1+1}(\Gamma) + (k', 0)$ does not intersect Θ and is smaller then it the order $\prec$. So, as

$$\Gamma^* \subset \widetilde{f}(\Gamma) + (k', 0) \;\Rightarrow\; \widetilde{f}^{a_1}(\Gamma^*) \cap \Theta = \emptyset \text{ and } \widetilde{f}^{a_1}(\Gamma^*) \prec \Theta.$$

The fact that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0, +\infty[\times[0,1]$ implies that $\widetilde{f}^{a_1}(\Gamma^*) \cap closure(\Omega)$ has an unlimited connected component, Γ^{**} which intersects w. So, Γ^{**} is not contained in B^- and thus there exists an integer $a_2 > 0$ such that $\widetilde{f}^{a_2}(\Gamma^{**}) \subset \widetilde{f}^{a_2+a_1+1}(\Gamma) + (k', 0)$ intersects $]0, +\infty[\times[0,1]$. In exactly the same way as above, we obtain an unlimited connected component of $\widetilde{f}^{a_2}(\Gamma^{**}) \cap closure(\Omega)$, denoted Γ^{***} which intersects w. So, Γ^{***} is not contained in B^- and there exists an integer $a_3 > 0$ such that $\widetilde{f}^{a_3}(\Gamma^{***})$ intersects $]0, +\infty[\times[0,1]$ and so on.

Thus, if we define $n_i = a_1 + a_2 + ... + a_i + 1$, we get that $n_i \overset{i\to\infty}{\to} \infty$ and for all $i \geq 1$, $\widetilde{f}^{n_i - 1}(\widetilde{f}(\Gamma) + (k', 0)) \supset \widetilde{f}^{n_i-1}(\Gamma^*) = \widetilde{f}^{a_2+\ldots+a_i}(\widetilde{f}^{a_1}(\Gamma^*)) \supset \widetilde{f}^{a_2+\ldots+a_i}(\Gamma^{**}) \supset ... \supset \widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*\ldots*}})$ and $\widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*\ldots*}})$ intersects $]0, +\infty[\times[0,1]$. So,

$$\widetilde{f}^{n_i}(\Gamma) \text{ intersects } V_0 - (k', 0) = V_{-k'}$$

and this contradicts theorem 2 and thus proves theorem 5.

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Figure captions.

Figure 1. Diagram showing $\widehat{A}$.

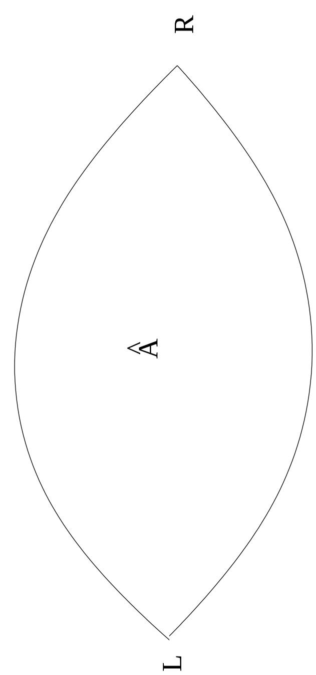
Figure 2. Diagram showing the set Γ_N.

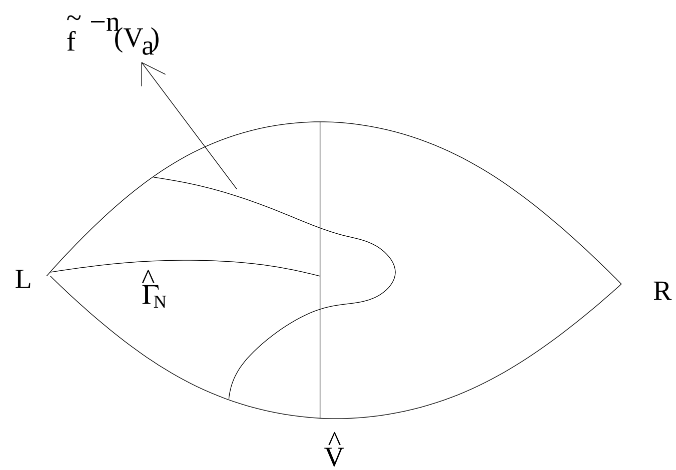
Figure 3. Diagram showing that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down} \cup \Gamma_{1a,up}$.

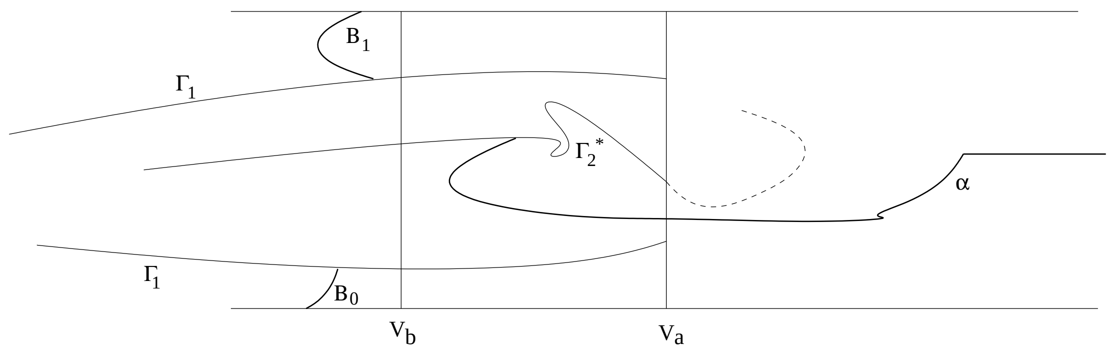
Figure 4. Diagram showing that either $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$ or $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$.

Figure 5. Diagram showing the sets Γ, $\Gamma - (1,0)$ and γ.

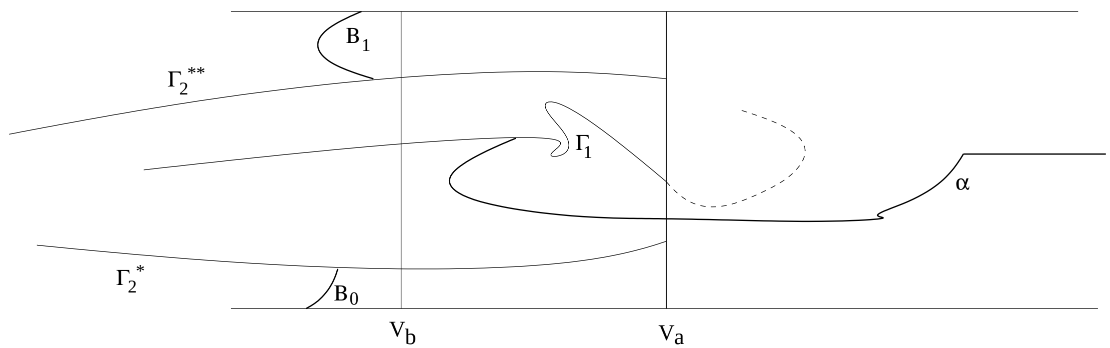
Figure 6. Diagram showing the sets $\overline{p(\Gamma)} \subset \gamma_E^-$.



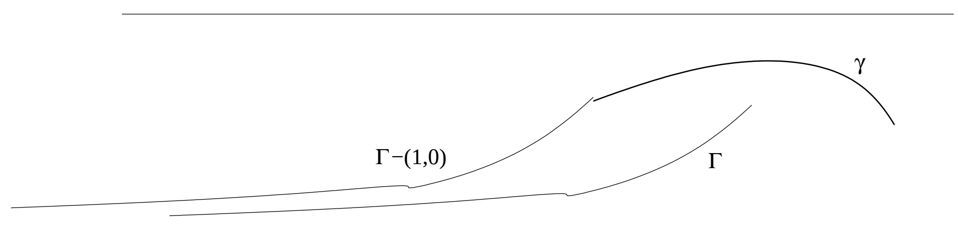

$\tilde{f}^{-n}(V_a)$
L
$\hat{\Gamma}_N$
R
$\hat{V}$

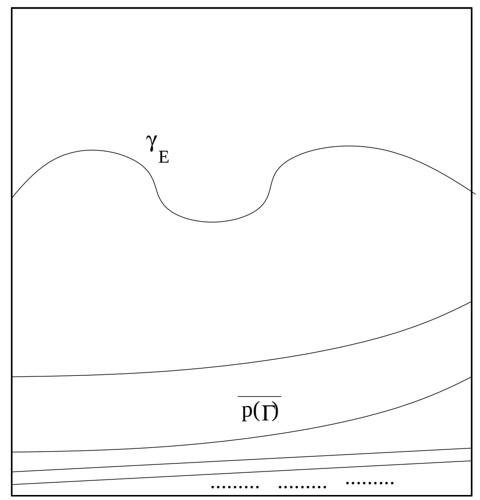


B_1
Γ_1
Γ_2^*
α
Γ_1
B_0
V_b
V_a



B1
Γ2**
Γ1
α
Γ2*
B0
Vb
Va


γ
Γ−(1,0)
Γ



γ_E
$\overline{p(\Gamma)}$